## Exercise 1:

Let $X \sim Logistic(\theta, 1)$, $\theta \in \mathbb{R}$

$$\Rightarrow f(x \| \theta) = \frac{e^{x-\theta}}{(1+e^{x-\theta})^2}, \quad x \in \mathbb{R}, \ \theta \in \mathbb{R}.$$

1)- Show that this family has an MLR property:

Let $\theta_1 < \theta_2$. We have

$$\frac{f(x\|\theta_2)}{f(x\|\theta_1)} = e^{\theta_1 - \theta_2}\left(\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}\right)^2.$$

Notice that $\frac{f(x\|\theta_2)}{f(x\|\theta_1)} \uparrow x \Leftrightarrow x \longmapsto \left(\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}\right)^2$ is nondecreasing.

$\Leftrightarrow x \overset{h}{\longmapsto} \frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}$ is nondecreasing,

since $\begin{cases} \theta_1 < \theta_2 \Rightarrow e^{x-\theta_2} < e^{x-\theta_1} \quad (*). \\ u \mapsto u^2 \uparrow \Leftrightarrow u > 0. \end{cases}$

Since $h'(x) = \frac{e^{x-\theta_1} - e^{x-\theta_2}}{(1+e^{x-\theta_2})^2} > 0$, by $(*)$,

then $\{Logistic(\theta, 1) : \theta \in \mathbb{R}\}$ has an MLR property in $x$.

2)- Find the MP-$\alpha$ test of $(H)$: $\begin{cases} H_0 : \theta = 0 \\ H_1 : \theta = 1. \end{cases}$

By the NP-lemma, the MP-$\alpha$ test of $(H)$ is

$$\varphi(x) = \begin{cases} 1, & \lambda(x) > k \\ \gamma, & \lambda(x) = k \\ 0, & \lambda(x) < k \end{cases}, \quad \text{with } \lambda(x) = \frac{f(x\|1)}{f(x\|0)}.$$

From 1), $x \mapsto \lambda(x)$ is nondecreasing. Further, X is continuous. Therefore, an MP test of (H) is:

$$\varphi(x) = \begin{cases} 1, & x > c \\ 0, & o/w \end{cases},$$

where c is determined via $P_{\theta=0}(X>c)=\alpha$.

We have $P_{\theta=0}(X>c) = \int_c^{\infty} \frac{e^{-t}dt}{(1+e^{-t})^2} = \left[-\frac{1}{1+e^{-t}}\right]_c^{\infty} = \frac{1}{1+e^{c}}$.

Therefore, $P_{\theta=0}(X>c) = \alpha \Leftrightarrow \frac{1}{1+e^{c}} = \alpha \Rightarrow c = \ln\left(\frac{1-\alpha}{\alpha}\right)$.

Thus, $\varphi(x) = \begin{cases} 1, & x > \ln\left(\frac{1-\alpha}{\alpha}\right) \\ 0, & o/w \end{cases}$ is MP-$\alpha$ test of (H).

For $\alpha = 0{,}2$, find the size of type-II error

If $\alpha = 0{,}2$, then $c = \ln\left(\frac{1-0{,}2}{0{,}2}\right) \approx 1{,}386$.

By definition, the size of type-II error is:

$$P(\text{type II error}) = 1 - P_{\theta=1}(X > 1{,}386) = P_{\theta=1}(X < 1{,}386)$$

$$= \int_{-\infty}^{1{,}386} \frac{e^{-(t-1)}dt}{(1+e^{-(t-1)})^2} = \left[-\frac{1}{1+e^{-(t-1)}}\right]_{-\infty}^{1{,}386} = \frac{e^{0{,}386}}{1+e^{0{,}386}} \approx 0{,}595.$$

3) - Show that the test in (2) is UMP-$\alpha$ for (H): $\begin{cases} H_0: \theta \le 0 \\ H_1: \theta > 0. \end{cases}$

Since $X \sim Logistic(\theta,1)$ and $\{Logistic(\theta,1) : \theta \in \mathbb{R}\}$ has an MLR property in $x$, then the Karlin-Rubin thm yields that $\varphi(x) = \begin{cases} 1, & x > \ln\left(\frac{1-\alpha}{\alpha}\right) \\ 0, & o/w \end{cases}$ is UMP-$\alpha$ for testing (H): $\begin{cases} H_0: \theta \le 0 \\ H_1: \theta > 0 \end{cases}$

## Exercise 2

Let $X_1, \dots, X_n \overset{iid}{\sim} \text{Pareto}(1, \theta)$ and $\mu = E(X)$.

$\Rightarrow f(x \| \theta) = \theta x^{-(\theta+1)}$, $x > 1$, $\theta > 1$.

1)- Express $\mu$ in terms of $\theta$:

By definition, $\mu = E(X) = \int_1^\infty x f(x) dx = \theta \int_1^\infty \frac{dx}{x^\theta}$

$$\Rightarrow \mu = E(X) = \theta \left[ -\frac{1}{(\theta-1)x^{\theta-1}} \right]_1^\infty = \frac{\theta}{\theta-1} \cdot (*)$$

2)- Find a UMP-$\alpha$ test of $(H): \begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu < \mu_0. \end{cases}$

We have $(H): \begin{cases} H_0: \frac{\theta}{\theta-1} = \frac{\theta_0}{\theta_0-1} \\ H_1: \frac{\theta}{\theta-1} < \frac{\theta_0}{\theta_0-1} \end{cases}$, by $(*)$.

$$\Leftrightarrow (H): \begin{cases} H_0: \theta = \theta_0 \\ H_1: \theta > \theta_0 \end{cases} \Leftrightarrow (H') \begin{cases} H_0: \theta = \theta_0 \\ H_1: \theta = \theta_1 \end{cases} \quad (\theta_1 > \theta_0).$$

The likelihood ratio is given by:

$$\lambda(\underline{x}) = \frac{L(\theta_1 \| \underline{x})}{L(\theta_0 \| \underline{x})} = \frac{\theta_1^n \prod_{i=1}^n x_i^{-(\theta_1+1)}}{\theta_0^n \prod_{i=1}^n x_i^{-(\theta_0+1)}} = \left(\frac{\theta_1}{\theta_0}\right)^n \left(\prod_{i=1}^n x_i\right)^{\theta_0-\theta_1} = \left(\frac{\theta_1}{\theta_0}\right)^n e^{(\theta_0-\theta_1)\sum_{i=1}^n \ln x_i}$$

By the NP-lemma, an MP-$\alpha$ test for $(H')$ is:

$$\varphi(\underline{x}) = \begin{cases} 1, & \lambda(\underline{x}) > k \\ \gamma, & \lambda(\underline{x}) = k \\ 0, & \lambda(\underline{x}) < k \end{cases} \quad (k > 0).$$

Since $\theta_0 - \theta_1 < 0$, then $\underline{x} \longmapsto \lambda(\underline{x})$ nonincreasing.

Therefore, an MP-$\alpha$ test for $(H')$ is

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Therefore, $\varphi(\underline{x}) = \begin{cases} 1, & \sum_{i=1}^{n} \ln x_i < c \\ 0, & \text{otherwise}, \end{cases}$ , as $\sum_{i=1}^{n} \ln X_i$ is a continuous r.v.

where c is determined via $E_{\theta_0}[\varphi(\underline{X})] = \alpha$. For this, we need to find the pdf of $\sum_{i=1}^{n} \ln X_i = T$.

Set $z = \ln x \Rightarrow x = e^{z} = \omega(z)$. The change of variable theorem yields that $g_Z(z) = f_X[\omega(z)]\,|\omega'(z)|$.

So, $g_Z(z) = \theta e^{-z(\theta+1)} e^{z} = \theta e^{-\theta z}$, $z > 0$.

Thus, $\ln X_i \sim Exp(\theta)$.

Hence, $\sum_{i=1}^{n} \ln X_i \sim \Gamma(n, \theta_0)$, when $H_0$ is true.

As such, c is $\theta_1$-free. As a result, the MP size $\alpha$ test $\varphi$ above does not depend on the choice of $\theta_1$, as long as $\theta > \theta_0$, ie it remains an MP test against any $\theta > \theta_0$. This is equivalent to concluding $\varphi$ is UMP-$\alpha$ test for (H).

## Exercise 3

Let $\Theta, \mathcal{X} \subset \mathbb{R}$. Suppose that $\mathcal{G} = \{g_\theta : \theta \in \Theta\}$ be a family of functions on $\mathcal{X}$ such that:

$$\begin{cases} \forall x \in \mathcal{X}, \ g_\theta(x) > 0. \\ \dfrac{\partial^2 \ln g_\theta(x)}{\partial\theta\,\partial x} < +\infty \end{cases}$$

1)- Show that $\mathcal{G}$ has MLR in $x$ is equivalent to one of the following conditions:

$$\begin{cases} i) \ \forall x \in \mathcal{X}, \theta \in \Theta, \ \dfrac{\partial^2}{\partial\theta\,\partial x} \ln g_\theta(x) \geq 0 \\ ii) \ \forall x \in \mathcal{X}, \theta \in \Theta, \ g_\theta(x) \dfrac{\partial^2}{\partial\theta\,\partial x} g_\theta(x) \geq \dfrac{\partial}{\partial\theta} g_\theta(x) \dfrac{\partial}{\partial x} g_\theta(x). \end{cases}$$

Since $\dfrac{\partial^2 \ln g_\theta(x)}{\partial\theta\,\partial x} < +\infty$, then $\forall x, \theta$ we have:

$$\frac{\partial^2 \ln g_\theta(x)}{\partial\theta\,\partial x} = \frac{\partial}{\partial\theta}\left[\frac{\partial \ln g_\theta(x)}{\partial x}\right] = \frac{\partial}{\partial\theta}\left[\frac{\frac{\partial g_\theta(x)}{\partial x}}{g_\theta(x)}\right]$$

$$= \frac{\left[\frac{\partial^2 g_\theta(x)}{\partial\theta\,\partial x}\right] g_\theta(x) - \left(\frac{\partial g_\theta(x)}{\partial\theta}\right)\left(\frac{\partial g_\theta(x)}{\partial x}\right)}{[g_\theta(x)]^2}$$

Therefore, $\dfrac{\partial^2 \ln g_\theta(x)}{\partial\theta\,\partial x} \geq 0 \iff \left(\dfrac{\partial^2 g_\theta(x)}{\partial\theta\,\partial x}\right) g_\theta(x) - \left(\dfrac{\partial g_\theta(x)}{\partial\theta}\right)\left(\dfrac{\partial g_\theta(x)}{\partial x}\right) \geq 0$,

since $\forall x \in \mathcal{X}$, $g_\theta(x) > 0$.

$$\iff \left(\frac{\partial^2 g_\theta(x)}{\partial\theta\,\partial x}\right) g_\theta(x) \geq \left(\frac{\partial}{\partial\theta} g_\theta(x)\right)\left(\frac{\partial}{\partial\theta} g_\theta(x)\right)$$

Thus, i) $\Leftrightarrow$ ii).

Hence in what follows, we prove that MLR property is equivalent to i).

Since $\forall x \in \mathcal{X}, \theta \in \Theta$, $\frac{\partial^2 \ln g_\theta(x)}{\partial\theta\partial x} = \frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial x}\ln g_\theta(x)\right] \geqslant 0$, by i)

then $\theta \longmapsto \frac{\partial}{\partial x}\ln g_\theta(x)$ is nondecreasing, $\forall x \in \mathcal{X}$.

This is equivalent to writing:

$$\theta_1 \leqslant \theta_2 \Longrightarrow \frac{\partial}{\partial x}\ln g_{\theta_1}(x) \leqslant \frac{\partial}{\partial x}\ln g_{\theta_2}(x)$$

$$\Rightarrow \frac{\partial}{\partial x}\left[\ln g_{\theta_2}(x) - \ln g_{\theta_1}(x)\right] \geqslant 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left[\ln\left(\frac{g_{\theta_2}(x)}{g_{\theta_1}(x)}\right)\right] \geqslant 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left[\frac{g_{\theta_2}(x)}{g_{\theta_1}(x)}\right]\frac{g_{\theta_1}(x)}{g_{\theta_2}(x)} \geqslant 0$$

$$\Leftrightarrow \frac{\partial}{\partial x}\left(\frac{g_{\theta_2}(x)}{g_{\theta_1}(x)}\right) \geqslant 0, \text{ as } \forall x \in \mathcal{X}, \theta \in \Theta, \ g_\theta(x) > 0.$$

$$\Rightarrow x \longmapsto \frac{g_{\theta_2}(x)}{g_{\theta_1}(x)} \text{ is nondecreasing}$$

Therefore, i) $\Leftrightarrow$ $\{g_\theta : \theta \in \Theta\}$ has an MLR property in $x$.

2)- Assume that $X \sim \mathcal{N}(\sqrt{\theta}, 1) \Rightarrow X^2 \sim \chi_1^2(\theta)$ (Non-central $\chi^2$-dbn)

$$\Rightarrow g_\theta(x) = \frac{1}{2\sqrt{2\pi x}}\left[e^{-\frac{(\sqrt{x}-\theta)^2}{2}} + e^{-\frac{(\sqrt{x}+\theta)^2}{2}}\right]$$

<u>Show that</u> $\{\chi_1^2(\theta)\}$ <u>has an MLR in $x$</u>:

Let $0 \leq \theta_1 < \theta_2$. The likelihood ratio is given by

$$\frac{g_{\theta_2}(x)}{g_{\theta_1}(x)} = \frac{e^{-\frac{(\sqrt{x}-\theta_2)^2}{2}} + e^{-\frac{(\sqrt{x}+\theta_2)^2}{2}}}{e^{-\frac{(\sqrt{x}-\theta_1)^2}{2}} + e^{-\frac{(\sqrt{x}+\theta_1)^2}{2}}} = \frac{e^{-\frac{\theta_2^2}{2}}\left(e^{\theta_2\sqrt{x}} + e^{-\theta_2\sqrt{x}}\right)}{e^{-\frac{\theta_1^2}{2}}\left(e^{\theta_1\sqrt{x}} + e^{-\theta_1\sqrt{x}}\right)}$$

$$= e^{\frac{1}{2}(\theta_1^2-\theta_2^2)}\frac{\cosh(\theta_2\sqrt{x})}{\cosh(\theta_1\sqrt{x})}, \text{ as } \cosh u = \frac{e^u + e^{-u}}{2}.$$

Set $\forall \theta, x \geq 0$, $h_\theta(x) = \cosh(\theta x)$. Notice that $\{g_\theta : \theta \geq 0\}$ has an MLR in $x \Leftrightarrow \{h_\theta : \theta \geq 0\}$ does. Since,

$$\begin{cases} \frac{\partial h_\theta(x)}{\partial x} = \theta \sinh(\theta x), \quad \frac{\partial h_\theta(x)}{\partial \theta} = x \sinh(\theta x), \text{ and} \\ \frac{\partial^2 h_\theta(x)}{\partial\theta\,\partial x} = \sinh(\theta x) + x\theta\cosh(\theta x), \text{ as } \begin{cases} (\cosh u)' = u'\sinh u \\ (\sinh u)' = u'\cosh u. \end{cases} \end{cases}$$

Then, $h_\theta(x)\frac{\partial^2 h_\theta(x)}{\partial\theta\,\partial x} = \cosh(\theta x)\sinh(\theta x) + x\theta\cosh^2(\theta x)$

$$= \frac{1}{2}\sinh(2\theta x) + x\theta\cosh^2(\theta x),$$

$$\geq x\theta\cosh^2(\theta x), \text{ as } \forall u \geq 0, \sinh u \geq 0.$$

$$\geq x\theta\sinh^2(\theta x), \text{ as } \forall u \geq 0, \cosh u \geq \sinh u$$

$$= \frac{\partial}{\partial x}h_\theta(x)\frac{\partial}{\partial\theta}h_\theta(x)$$

Therefore, $\{h_\theta : \theta \geq 0\}$ has an MLR in $x$, by 1).

But $x \mapsto \sqrt{x}$ is nondecreasing. Thus,

$\{g_\theta \,/\, \forall \theta, x \geq 0,\ g_\theta(x) = h_\theta(\sqrt{x})\}$ has an MLR in $x$.

## Exercise 4

Let $X$ be a random variable whose pmf under $H_i$, $i=\overline{1,2}$ is given by:

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $f_0(x)$ | 0,01 | 0,01 | 0,01 | 0.01 | 0.01 | 0,01 | 0,94 |
| $f_1(x)$ | 0.06 | 0.05 | 0.04 | 0.03 | 0.02 | 0.01 | 0,79 |

1)- Find the MP test for $H_0$ vs $H_1$ with size $\alpha = 0.04$

The likelihood ratio $\lambda(x) = \frac{f_1(x)}{f_0(x)}$ is given by:

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\lambda(x)$ | 6 | 5 | 4 | 3 | 2 | 1 | 0,84 |

By the N-P lemma, the MP-0,04 test of $H_0$ vs $H_1$ is:

$$\varphi(x) = \begin{cases} 1, & \lambda(x) > k \\ \gamma, & \lambda(x) = k \\ 0, & \lambda(x) < k. \end{cases}$$

Notice that $x \longmapsto \lambda(x)$ is nonincreasing.

Therefore, the MP test of $H_0$ vs $H_1$ is

$$\varphi(x) = \begin{cases} 1, & x < c \\ \gamma, & x = c \\ 0, & x > c, \end{cases}$$

with $E[\varphi(X)] = \alpha$, under $H_0$, that is:

$$P(X<c) + \gamma P(X=c) = 0,04. \text{ when } H_0 \text{ is true.}$$

Using the table of distribution of $X$ when $H_0$ is true, this size condition is fulfilled for $c=4$ and $\gamma=1$.

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Thus, the MP 0,09 test for $H_0$ vs $H_1$ is

$$\varphi(x) = \begin{cases} 1, & x \leq 4 \\ 0, & o/w. \end{cases}$$

2)- <u>Compute the probability of type II error for $\varphi$:</u>

By definition, $P(\text{type-II error}) = 1 - P(X \leq 4)$, when $H_1$ is true.

$$= 1 - (0{,}06 + 0{,}05 + 0{,}04 + 0{,}03)$$
$$= 1 - 0{,}18$$
$$= 0{,}82 .$$

Thus, $P(\text{type-II error}) = 0{,}82$ .

## Exercise 5:

Let $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{U}(\theta, \theta+1)$. Consider testing (H): $\begin{cases} H_0 : \theta = 0 \\ H_1 : \theta > 0. \end{cases}$

Assume that $\varphi(\underline{x}) = \begin{cases} 1, & x_{(n)} \geq 1 \text{ or } x_{(1)} \geq k \\ 0, & \text{otherwise.} \end{cases}$

1)- Find the constant $k$ that makes $\varphi$ a size $\alpha$ test.

$\varphi$ is a size $\alpha$ test if $E_0[\varphi(\underline{X})] = \alpha$.

$$\Leftrightarrow P_0[(X_{(n)} \geq 1) \cup (X_{(1)} \geq k)] = \alpha.$$

$$\Leftrightarrow P_0(X_{(n)} \geq 1) + P_0(X_{(1)} \geq k) - P_0[(X_{(n)} \geq 1) \cap (X_{(1)} \geq k)] = \alpha. \quad (*)$$

Notice that under $H_0 : \theta = 0$, $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{U}(0,1)$.

By monotonicity of $P_0$, it holds that:

$$P_0[(X_{(n)} \geq 1) \cap (X_{(1)} \geq k)] \leq P[X_{(n)} \geq 1) = 1 - P(X_{(n)} < 1) = 1 - 1^n = 0$$

Therefore $(*)$ reduces to $P_0(X_{(1)} \geq k) = \alpha$

$$\Leftrightarrow (1-k)^n = \alpha, \text{ for } k \in (0,1]$$

$$\Rightarrow k = 1 - \sqrt[n]{\alpha}.$$

2)- Find the power function of $\varphi$:

By definition, $\beta_\varphi(\theta) = P_\theta[(X_{(n)} \geq 1) \cup (X_{(1)} \geq k)]$, $\forall \theta$

$$= P_\theta(X_{(n)} \geq 1) + P_\theta(X_{(1)} \geq k) - P_\theta((X_{(n)} \geq 1) \cap (X_{(1)} \geq k))$$

Since $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{U}(\theta, \theta+1)$, then:

$*$ $P_\theta(X_{(n)} \geq 1) = \begin{cases} 0, & \theta < 0 \\ 1 - (1-\theta)^n, & 0 \leq \theta \leq 1 \\ 1, & \theta > 1 \end{cases}$

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* $P_\theta(X_{(1)} > k) = \begin{cases} 0 & , \theta+1<k \\ (\theta+1-k)^n, & \theta \le k \le \theta+1 \\ 1 & , k<\theta \end{cases} = \begin{cases} 0, & \theta<k-1 \\ (\theta+1-k)^n, & k-1 \le \theta \le k \\ 1, & \theta>k. \end{cases}$

* To compute $P_\theta[(X_{(1)} \ge k) \cap (X_{(n)} \ge 1)]$, recall that the joint pdf of $(X_{(1)}, X_{(n)})$ is

$$f_{(X_{(1)},X_{(n)})}(x,y|\theta) = n(n-1)(y-x)^{n-2} \mathbb{1}_{(\theta<x<y<\theta+1)},$$

So, $P_\theta[(X_{(1)} \ge k) \cap (X_{(n)} \ge 1)] = \begin{cases} 0 & , \theta+1<1 \\ n(n-1)\int_1^{\theta+1}\left(\int_k^y (y-x)^{n-2}\,dx\right)dy & , \theta<k, \theta+1>1 \\ 1 & , k \le \theta, 1 \le \theta \end{cases}$

$$= \begin{cases} 0 & , \theta<0 \\ (\theta+1-k)^n-(1-k)^n, & 0 \le \theta \le k \\ 1 & , \theta>k \end{cases}$$

Therefore, the power function of $\varphi$ is:

$$\beta_\varphi(\theta) = \begin{cases} 0 & , \theta<k-1 \\ (\theta+1-k)^n & , k-1 \le \theta \le 0 \\ 1-(1-\theta)^n-(1-k)^n & , 0<\theta \le k \\ 1 & , k<\theta \end{cases} = \begin{cases} 0 & , \theta<-\sqrt[n]{\alpha} \\ (\theta+\alpha^{\frac{1}{n}})^n & , -\sqrt[n]{\alpha} \le \theta \le 0 \\ 1+\alpha-(1-\theta)^n, & 0<\theta \le 1-\sqrt[n]{\alpha} \\ 1 & , \theta>1-\sqrt[n]{\alpha}. \end{cases}$$

3)- Show that $\varphi$ is a UMP size $\alpha$ test:

Notice that $(H): \begin{cases} H_0: \theta=0 \\ H_1: \theta>0 \end{cases} \Leftrightarrow (H') \begin{cases} H_0: \theta=0 \\ H_1: \theta=\theta_1, \end{cases} (\theta_1>0).$

The likelihood ratio is given by:

$$\lambda(x,y)=\frac{f_{(X_{(1)},X_{(n)})}(x,y|\theta)}{f_{(X_{(1)},X_{(n)})}(x,y\|0)}=\begin{cases}0, & 0\leq x<\theta_1,\ y\leq 1\\ 1, & \theta_1<x\leq y\leq 1\\ \infty, & y>1\end{cases}$$

$\forall\theta_1\in(0,1-\sqrt[n]{\alpha}]$, the N-P lemma yields that

$$\varphi_1(x,y)=\begin{cases}1, & \lambda(x,y)>k\\ \gamma, & \lambda(x,y)=k\\ 0, & \lambda(x,y)<k.\end{cases}$$

is the MP test for $(H')$: $\begin{cases}H_0:\theta=0\\ H_1:\theta=\theta_1,\end{cases}$

where $E_0[\varphi(X_{(1)},X_{(n)})]=\alpha \Leftrightarrow P_0(\lambda(x,y)>k)+\gamma P_0(\lambda(x,y)=k)=\alpha$ (**)

Notice that (**) holds for $k=1$, that is:

$$P_0(X_{(n)}>1)+\gamma P(\theta_1<X_{(1)}<X_{(n)}\leq 1)=\alpha$$

$$\Rightarrow\quad 0+\gamma(1-\theta_1)^n=\alpha$$

$$\Rightarrow\quad \gamma=\frac{\alpha}{(1-\theta_1)^n}$$

Thus, $\varphi_1(x_{(1)},x_{(n)})=\begin{cases}1, & x_{(n)}>1\\ \frac{\alpha}{(1-\theta_1)^n}, & \theta_1<x_{(1)}<x_{(n)}<1\\ 0, & o/w.\end{cases}$

is UMP for $(H)$.

4)- Find the values of $n$ and $k$ / $\varphi$ rejects $H_0$ if $X_{(n)}\geq 1$ or $X_{(1)}\geq k$ at level 0,1 will have the power at least 0,8 if $\theta>1$.

Since $\beta(\theta)=1$, if $\theta>1$, then for $k=1-(0{,}1)^{\frac{1}{n}}$ and any sample size $n$, the power of the test is 1 (and hence at least 0,8) if $\theta>1$.

## Exercises:

Let $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(\mu_1, \sigma_1^2)$

$Y_1, \dots, Y_m \overset{iid}{\sim} \mathcal{N}(\mu_2, \sigma_2^2)$

Consider testing $(H): \begin{cases} H_0: \mu_1 = \mu_2 \\ H_1: \mu_1 \neq \mu_2 \end{cases}$ with $\sigma_1^2 = \sigma_2^2 = \sigma^2$.

1)- Find the LRT of $(H)$:

For notational convenience, let $\theta = (\mu_1, \mu_2, \sigma^2)$.

- By definition, $L(\theta \| \underline{x}, \underline{y}) = \left(\prod_{i=1}^{n} f(x_i \| \theta)\right)\left(\prod_{j=1}^{m} f(y_j \| \theta)\right)$, as $X \perp Y$.

$$\Rightarrow L(\theta \| \underline{x}, \underline{y}) = \frac{1}{(2\pi\sigma^2)^{\frac{n+m}{2}}} e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i - \mu_1)^2 + \sum_{j=1}^{m}(y_j - \mu_2)^2\right]}$$

$$\Rightarrow \ell(\theta \| \underline{x}, \underline{y}) = \ln L(\theta \| \underline{x}, \underline{y})$$

$$\Rightarrow \ell(\theta | \underline{x}, \underline{y}) = -\frac{(n+m)}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i - \mu_1)^2 + \sum_{j=1}^{m}(y_j - \mu_2)^2\right] \quad (*)$$

$$\Rightarrow \begin{cases} \frac{\partial \ell}{\partial \mu_1} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu_1) \\ \frac{\partial \ell}{\partial \mu_2} = \frac{1}{\sigma^2}\sum_{j=1}^{m}(y_j - \mu_2) \\ \frac{\partial \ell}{\partial \sigma^2} = -\frac{(n+m)}{2\sigma^2} + \frac{1}{2\sigma^4}\left[\sum_{i=1}^{n}(x_i - \mu_1)^2 + \sum_{j=1}^{m}(y_j - \mu_2)^2\right]. \end{cases}$$

Now $\begin{cases} \frac{\partial \ell}{\partial \mu_1} = 0 \\ \frac{\partial \ell}{\partial \mu_2} = 0 \\ \frac{\partial \ell}{\partial \sigma^2} = 0 \end{cases} \Rightarrow \begin{cases} \hat{\mu}_1 = \bar{x} \\ \hat{\mu}_2 = \bar{y} \\ \hat{\sigma}^2 = \frac{1}{n+m}\left[\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_j - \bar{y})^2\right]. \end{cases}$

It remains to show that $\theta \longmapsto \ell(\theta \| \underline{x}, \underline{y})$ attains its maximum at $\hat{\theta} = (\bar{x}, \bar{y}, \hat{\sigma}^2)$. For this,

$$Hl(\hat{\theta} \| x, y) = \begin{pmatrix} \frac{\partial^2 l(\hat{\theta})}{\partial \mu_1^2} & \frac{\partial^2 l(\hat{\theta})}{\partial \mu_1 \partial \mu_2} & \frac{\partial^2 l(\hat{\theta})}{\partial \mu_1 \partial \sigma^2} \\ \frac{\partial^2 l(\hat{\theta})}{\partial \mu_2 \partial \mu_1} & \frac{\partial^2 l(\hat{\theta})}{\partial \mu_2^2} & \frac{\partial^2 l(\hat{\theta})}{\partial \mu_2 \partial \sigma^2} \\ \frac{\partial^2 l(\hat{\theta})}{\partial \sigma^2 \partial \mu_1} & \frac{\partial^2 l(\hat{\theta})}{\partial \sigma^2 \partial \mu_2} & \frac{\partial^2 l(\hat{\theta})}{\partial (\sigma^2)^2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{n}{\hat{\sigma}^2} & 0 & 0 \\ 0 & -\frac{m}{\hat{\sigma}^2} & 0 \\ 0 & 0 & -\frac{(n+m)}{\hat{\sigma}^4} \end{pmatrix}$$

Since the Hessian matrix $Hl(\hat{\theta} \| x, y)$ is diagonal, then $-\frac{n}{\hat{\sigma}^2}$, $-\frac{m}{\hat{\sigma}^2}$ and $-\frac{(n+m)}{\hat{\sigma}^4}$ are the eigenvalues of $Hl(\hat{\theta} \| x, y)$. Since they are all nonpositive, then the quadratic form $Q$ associated with $Hl(\hat{\theta} \| x, y)$ is negative definite. Therefore, $\theta \mapsto l(\theta \| x, y)$ attains its maximum at $\hat{\theta} = (\bar{x}, \bar{y}, \hat{\sigma}^2)$. Thus, $\bar{X}$, $\bar{Y}$ and $\hat{\sigma}^2$ are respectively the MLEs for $\mu_1$, $\mu_2$ and $\sigma^2$.

- Under $H_0 : \mu_1 = \mu_2 = \mu$, the log-likelihood $(*)$ becomes

$$l(\mu, \sigma^2 \| x, y) = -\frac{(n+m)}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \left[ \sum_{i=1}^{n} (x_i - \mu)^2 + \sum_{j=1}^{m} (y_j - \mu)^2 \right]$$

$$\Rightarrow \begin{cases} \frac{\partial l}{\partial \mu} = \frac{1}{\sigma^2} \left[ \sum_{i=1}^{n} (x_i - \mu)^2 + \sum_{j=1}^{m} (y_j - \mu)^2 \right] \\ \frac{\partial l}{\partial \sigma^2} = -\frac{(n+m)}{2\sigma^2} + \frac{1}{2\sigma^4} \left[ \sum_{i=1}^{n} (x_i - \mu)^2 + \sum_{j=1}^{m} (y_j - \mu)^2 \right]. \end{cases}$$

$$\text{for } \begin{cases} \frac{\partial l}{\partial \mu} = 0 \\ \frac{\partial l}{\partial \sigma^2} = 0 \end{cases} \Rightarrow \begin{cases} \hat{\mu}_0 = \frac{1}{n+m}\left[\sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right] = \frac{n\bar{x}+m\bar{y}}{n+m} \\ \hat{\sigma}_0^2 = \frac{1}{n+m}\left[\sum_{i=1}^n (x_i-\hat{\mu}_0)^2 + \sum_{j=1}^m (y_j-\hat{\mu}_0)^2\right] \end{cases}$$

We are left to prove that $(\mu,\sigma^2) \mapsto l(\mu,\sigma^2 \| \underline{x},\underline{y})$ attains its maximum at $(\hat{\mu}_0, \hat{\sigma}_0^2)$. Since

$$Hl(\hat{\mu}_0, \hat{\sigma}_0^2 \| \underline{x},\underline{y}) = \begin{pmatrix} -\frac{(n+m)}{\hat{\sigma}^2} & 0 \\ 0 & -\frac{(n+m)}{2\hat{\sigma}^4} \end{pmatrix},$$

then $Hl(\hat{\mu}_0, \hat{\sigma}_0^2 \| \underline{x},\underline{y})$ has negative eigenvalues $-\frac{(n+m)}{\hat{\sigma}^2}$ and $-\frac{(n+m)}{2\hat{\sigma}^4}$. Therefore, the quadratic form associated with $Hl(\hat{\mu}_0, \hat{\sigma}_0^2 \| \underline{x},\underline{y})$ is negative definite. Thus, $(\mu,\sigma^2) \mapsto l(\mu,\sigma^2 \| \underline{x},\underline{y})$ attains its maximum at $(\hat{\mu}_0, \hat{\sigma}_0^2)$. Hence, the pooled estimators $\hat{\mu}_0$ and $\hat{\sigma}_0^2$ are MLEs for $\mu$ and $\sigma^2$ when $H_0$ is true.

- It follows that the LRT statistic is:

$$\lambda(\underline{x},\underline{y}) = \frac{L(\hat{\mu}_0, \hat{\mu}_0, \hat{\sigma}_0^2 \| \underline{x},\underline{y})}{L(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}^2 \| \underline{x},\underline{y})} = \left(\frac{2\pi\hat{\sigma}_0^2}{2\pi\hat{\sigma}^2}\right)^{\frac{m+n}{2}} \frac{e^{-\frac{1}{2\hat{\sigma}_0^2}\left[\sum_{i=1}^n (x_i-\hat{\mu}_0)^2 + \sum_{j=1}^m (y_j-\hat{\mu}_0)^2\right]}}{e^{-\frac{1}{2\hat{\sigma}^2}\left[\sum_{i=1}^n (x_i-\bar{x})^2 + \sum_{j=1}^m (y_j-\bar{y})^2\right]}}$$

$$= \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{\frac{m+n}{2}} e^{-\frac{(n+m)}{2} + \frac{(n+m)}{2}} = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{\frac{m+n}{2}}$$

- Thus, a size $\alpha$ LRT test for $(H): \begin{cases} H_0: \mu_1 = \mu_2 \\ H_1: \mu_1 \neq \mu_2 \end{cases}$ is

$$\varphi(\underline{x},\underline{y}) = \begin{cases} 1, & \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{\frac{m+n}{2}} < k \\ 0, & \text{otherwise.} \end{cases}$$

Show that the LRT in part (1) can be based on the statistic: $T = \dfrac{\bar{X} - \bar{Y}}{\sqrt{S_p^2\left(\frac{1}{n} + \frac{1}{m}\right)}}$, where

$$S_p^2 = \frac{1}{n+m-2}\left[\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{j=1}^{m}(Y_j - \bar{Y})^2\right].$$

Recalling that $\lambda(\underline{x}, \underline{y}) = \left(\dfrac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{\frac{n+m}{2}} = \left[\dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_j - \bar{y})^2}{\sum_{i=1}^{n}(x_i - \hat{\mu}_0)^2 + \sum_{j=1}^{m}(y_j - \hat{\mu}_0)^2}\right]^{\frac{n+m}{2}}$

Notice that:

$$* \sum_{i=1}^{n}(x_i - \hat{\mu}_0)^2 = \sum_{i=1}^{n}\left(x_i - \frac{n\bar{x} + m\bar{y}}{n+m}\right)^2 = \sum_{i=1}^{n}\left[(x_i - \bar{x}) + \left(\bar{x} - \frac{n\bar{x} + m\bar{y}}{n+m}\right)\right]^2$$

$$= \sum_{i=1}^{n}(x_i - \bar{x})^2 + \frac{nm^2}{(n+m)^2}(\bar{x} - \bar{y})^2.$$

$*$ Analogously, $\sum_{j=1}^{m}(y_j - \hat{\mu}_0)^2 = \sum_{j=1}^{m}(y_j - \bar{y})^2 + \dfrac{mn^2}{(n+m)^2}(\bar{x} - \bar{y})^2.$

Therefore, $\sum_{i=1}^{n}(x_i - \hat{\mu}_0)^2 + \sum_{j=1}^{m}(y_j - \hat{\mu}_0)^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_j - \bar{y})^2 + \dfrac{nm}{n+m}(\bar{x} - \bar{y})^2.$

Thus, $$\lambda(\underline{x}, \underline{y}) = \frac{1}{\left[1 + \dfrac{nm(\bar{x} - \bar{y})^2}{(n+m)\left[\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_j - \bar{y})^2\right]}\right]^{\frac{n+m}{2}}}.$$

But $\lambda$ is a nonincreasing function of $\dfrac{nm(\bar{x} - \bar{y})^2}{(n+m)\left[\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_j - \bar{y})^2\right]}$

Hence, a size $\alpha$ LRT test $\varphi$ rejects $H_0: \mu_1 = \mu_2$ if $\lambda(\underline{x}, \underline{y}) < k$, $k \in (0,1)$.

$$\Leftrightarrow \frac{(\bar{x} - \bar{y})^2}{\frac{n+m}{nm}\left[(n-1)S_1^2 + (m-1)S_2^2\right]} > c.$$

$$\Leftrightarrow \quad \frac{(\bar{x}-\bar{y})^2}{\left(\frac{1}{n}+\frac{1}{m}\right)(n+m-2)S_p^2} > c$$

$$\Leftrightarrow \quad \frac{(\bar{x}-\bar{y})^2}{\left(\frac{1}{n}+\frac{1}{m}\right)S_p^2} > c_1$$

$$\Leftrightarrow \quad \left|\frac{\bar{x}-\bar{y}}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}}\right| > c_2$$

$$\Leftrightarrow \quad |t| > c_2$$

Finally, a size $\alpha$ LRT test is $\varphi(t)=\begin{cases}1, & |t|>c_2.\\ 0, & o/w.\end{cases}$

3) - <u>Find the distribution of $T$ under $H_0$:</u>

By definition, $T=\dfrac{\bar{X}-\bar{Y}}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}}$

Under $H_0: \mu_1=\mu_2=\mu$, $\bar{X}-\bar{Y} \sim \mathcal{N}\left(0, \sigma^2\left(\frac{1}{n}+\frac{1}{m}\right)\right)$

$$\Rightarrow \frac{\bar{X}-\bar{Y}}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}} \sim \mathcal{N}(0,1).$$

Further, $\dfrac{(n-1)S_1^2}{\sigma^2} \sim \chi^2_{n-1}$ and $\dfrac{(m-1)S_2^2}{\sigma^2} \sim \chi^2_{m-1}$.

Since $\sigma_1^2=\sigma_2^2=\sigma^2$ and $S_1^2 \perp\!\!\!\perp S_2^2$, then $\dfrac{(n-1)S_1^2}{\sigma^2}+\dfrac{(m-1)S_2^2}{\sigma^2}=\dfrac{(n+m-2)S_p^2}{\sigma^2}$

As a result, $\dfrac{(n+m-2)}{\sigma^2}S_p^2 \sim \chi^2_{n+m-2}$.

Thus,

$$\frac{\dfrac{\bar{X}-\bar{Y}}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}}}{\sqrt{\dfrac{\frac{(n+m-2)S_p^2}{\sigma^2}}{n+m-2}}} \sim T_{n+m-2}, \quad \text{as} \begin{cases} (\bar{X}-\bar{Y}) \perp\!\!\!\perp S_p^2. \\ \text{In addition, if } U\sim\mathcal{N}(0,1),\ V\sim\chi^2_\nu \\ \text{and } U \perp\!\!\!\perp V, \text{ then} \\ T=\dfrac{U}{\sqrt{V/\nu}} \sim T_\nu. \end{cases}$$