Exercise 2:

Let $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$, $\lambda, \mu > 0$ and $X \perp\!\!\!\perp Y$.

Consider testing $(H): \begin{cases} H_0: \mu \geq \lambda \\ H_1: \mu < \lambda. \end{cases}$

Find a UMPI-α test of (H):

- Consider the scale group:

$$\mathcal{G} = \left\{ g_c \in \mathcal{F}(\mathbb{R}_+^{2*}, \mathbb{R}_+^{2*}) \,/\, g_c : \begin{array}{l} \mathbb{R}_+^{2*} \longrightarrow \mathbb{R}_+^{2*} \\ (x,y) \longmapsto (cx, cy) \end{array} \right\}, \text{ where } c>0.$$

- Since $X \perp\!\!\!\perp Y$, then the joint pdf of (X,Y) is:

$$f_{(X,Y)}(x,y \,\|\, \lambda,\mu) = \lambda\mu e^{-(\lambda x + \mu y)}, \quad x,y>0$$

But $\forall c>0$, $g_c \in \mathcal{G}$, $g_c(X,Y) = (cX, cY) = (U,V)$.

Therefore, $\begin{cases} u = cx \\ v = cy \end{cases} \Rightarrow \begin{cases} x = \frac{u}{c} \\ y = \frac{v}{c} \end{cases}$

$$\rightarrow J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{c} & 0 \\ 0 & \frac{1}{c} \end{vmatrix} = \frac{1}{c^2}$$

Thus, the joint pdf of (U,V) is:

$$h_{(U,V)}(u,v) = f_{(X,Y)}\left(\frac{u}{c}, \frac{v}{c}\right)|J| = \frac{\lambda\mu}{c^2} e^{-\frac{1}{c}(\lambda u + \mu v)}, \quad u,v>0$$

$$= \left(\frac{\lambda}{c} e^{-\frac{\lambda}{c}u}\right)\left(\frac{\mu}{c} e^{-\frac{\mu}{c}v}\right) = f_{(X,Y)}\left(u,v \,\|\, \frac{\lambda}{c}, \frac{\mu}{c}\right)$$

Hence, $\forall g_c \in \mathcal{G}$, $\theta = (\lambda,\mu) \in \mathbb{R}_+^{2*}$, $\exists!\, \theta' = \left(\frac{\lambda}{c}, \frac{\mu}{c}\right) \in \mathbb{R}_+^{2*}$ such that:

$$(X,Y) \sim f(x,y \,\|\, \theta) \Longrightarrow g_c(X,Y) \sim f(x,y \,\|\, \theta').$$

* Under $H_0: \mu \geq \lambda \Leftrightarrow \frac{\mu}{c} \geq \frac{\lambda}{c}$, $\forall c>0$
Under $H_1: \mu < \lambda \Leftrightarrow \frac{\mu}{c} < \frac{\lambda}{c}$, $\forall c>0$.
Thus H_0 and H_1 are invariant under $\mathcal{G}$.
So, is the testing problem (H).

- Now, consider $T_\beta(x,y) = \left(\frac{x}{\beta}, \frac{y}{\beta}\right)$, where $\beta = \sqrt{x^2+y^2}$.

* Since $\forall (x,y) \in \mathbb{R}_+^{2*}$ and $g_c \in \mathcal{G}$, we have

$$T_\beta[g_c(x,y)] = T_\beta(cx, cy) = \left(\frac{cx}{c\beta}, \frac{cy}{c\beta}\right) = \left(\frac{x}{\beta}, \frac{y}{\beta}\right) = T_\beta(x,y).$$

Then T_β is invariant under $\mathcal{G}$.

* On the other hand, assume that $\forall (x,y), (x',y') \in \mathbb{R}_+^{2*}$,

$$T_\beta(x,y) = T_{\beta}(x',y') \Leftrightarrow \left(\frac{x}{\beta}, \frac{y}{\beta}\right) = \left(\frac{x'}{\beta'}, \frac{y'}{\beta'}\right), \text{ with } \beta' = \sqrt{x'^2+y'^2}$$

$$\Rightarrow \begin{cases} x = \frac{\beta}{\beta'} x' \\ y = \frac{\beta}{\beta'} y' \end{cases}$$

$$\Rightarrow (x,y) = \left(\frac{\beta}{\beta'} x', \frac{\beta}{\beta'} y'\right) = g_{\frac{\beta}{\beta'}}(x',y'), \ \frac{\beta}{\beta'} > 0.$$

$$\Rightarrow \exists\, g_{\beta/\beta'} \in \mathcal{G} \,/\, (x,y) = g_{\beta/\beta'}(x',y').$$

Thus, T_β is maximal under $\mathcal{G}$.
Hence, any decision rule ϕ of (H) is invariant $\Leftrightarrow \phi = h(T_\beta)$. The problem of finding a UMPI test of (H) boils down to finding a UMP test of (H) based on $T_\beta(X,Y)$.

* Notice that $\frac{X}{Y} = \frac{X/\beta}{Y/\beta} = h[T_\beta(X,Y)]$
Likewise, $\frac{Y}{X} = \frac{Y/\beta}{X/\beta} = h[T_\beta(X,Y)]$.

$\Bigg\}$ One can choose to work with one of these functions of $T_\beta(X,Y)$.

- Next, we find the pdf of $\frac{X}{Y}$. For this, set

$$\begin{cases} S = \frac{X}{Y} \\ W = Y \end{cases} \Rightarrow \begin{cases} s = \frac{x}{y} \\ w = y \end{cases} \Rightarrow \begin{cases} x = sw \\ y = w \end{cases}$$

$$\Rightarrow J(s,w) = \begin{vmatrix} \frac{dx}{ds} & \frac{dx}{dw} \\ \frac{dy}{ds} & \frac{dy}{dw} \end{vmatrix} = \begin{vmatrix} w & s \\ 0 & 1 \end{vmatrix} = w$$

Therefore, the joint pdf of (S,W) is:

$$g_{(S,W)}(s,w) = f_{(X,Y)}(sw, w)|J| \quad , \quad s,w>0$$

Thus, the marginal pdf of S is:

$$g_S(s) = \int_0^\infty g(s,w)\,dw = \lambda\mu \int_0^\infty e^{-(\lambda s w + \mu w)} w\,dw$$

By integration by part, we get:

$$g_S(s) = \lambda\mu \left[\frac{-w e^{-(\lambda s w + \mu w)}}{\lambda s + \mu} \right]_0^\infty + \frac{\lambda\mu}{\lambda s + \mu} \int_0^\infty e^{-(\lambda s + \mu) w}\,dw$$

$$= \frac{\lambda\mu}{(\lambda s + \mu)^2} = \frac{\mu/\lambda}{(\mu/\lambda + s)^2}, \quad s>0.$$

- By setting $\eta = \frac{\mu}{\lambda} > 0$, the hypothesis testing problem (H) is equivalent to: $\begin{cases} H_0 : \eta \geq 1 \\ H_1 : \eta < 1 \end{cases}$. Moreover,

$g_S(s) = \frac{\eta}{(\eta + s)^2}$, $s>0$. Now, $\forall \eta_1, \eta_2$ such that $\eta_2 > \eta_1$,

$$\frac{g_{\eta_2}(s)}{g_{\eta_1}(s)} = \frac{\eta_2}{\eta_1} \left(\frac{\eta_1 + s}{\eta_2 + s} \right)^2. \text{ But } \left(\frac{\eta_1 + s}{\eta_2 + s} \right)' = \frac{\eta_2 - \eta_1}{(\eta_2 + s)^2} > 0$$

Thus $s \longmapsto \frac{\eta_1 + s}{\eta_2 + s}$ is nondecreasing. So is $s \longmapsto g_{\eta_2}(s) / g_{\eta_1}(s)$, as $x \longmapsto x^2 \uparrow$ on $\mathbb{R}_+^*$, as $\frac{\eta_1 + s}{\eta_2 + s} > 0$.

Hence $\{g(s\|\eta) : \eta > 0\}$ has an MLR property in $S = \frac{X}{Y}$. As a result, the Karlin-Rubin theorem, a UMP test of size α is:

$$\varphi(s) = \begin{cases} 1, & s < k \\ 0, & o/w, \end{cases}$$

for some $k > 0$ such that $P_{\eta=1}(S < k) = \alpha$.

$$\Rightarrow \int_0^k \frac{ds}{(1+s)^2} = \alpha$$

$$\Rightarrow \left[-\frac{1}{1+s}\right]_0^k = \alpha$$

$$\Rightarrow \frac{k}{1+k} = \alpha$$

$$\Rightarrow k = \frac{\alpha}{1-\alpha}.$$

All in all $\varphi\left(\frac{x}{y}\right) = \begin{cases} 1, & \frac{x}{y} < \frac{\alpha}{1-\alpha} \\ 0, & o/w. \end{cases}$ is a UMPI test of (H).

Exercise 5:

Let $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(\theta, \sigma^2)$, σ^2 - known.

$\theta \sim Laplace(0, \tau)$, τ - known.

Consider testing $(H): \begin{cases} H_0 : \theta \leq 0 \\ H_1 : \theta > 0. \end{cases}$

1) Calculate $P(\theta > c \,\|\, \underline{x})$

For this, we need to find $\pi(\theta \| \underline{x})$.

Notice that $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a sufficient statistic for θ. Further, $\bar{X} \sim \mathcal{N}(\theta, \frac{\sigma^2}{n})$. Therefore,

$$\pi(\theta \| \underline{x}) = f_{\bar{X}}(\bar{x}|\theta)\,\pi(\theta)$$

$$= \left(\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2}\right)\left(\frac{1}{2\tau} e^{-\frac{|\theta|}{\tau}}\right)$$

$$\propto e^{-\frac{\theta^2 - 2\bar{x}\theta - 2\left(\frac{\sigma^2}{n\tau}\right)|\theta|}{2(\sigma^2/n)}} := g(\theta)$$

Thus, $\pi(\theta \| \underline{x}) = \dfrac{g(\theta)}{\int_{-\infty}^{\infty} g(\theta)\, d\theta}$.

Recall that $\int_{-\infty}^{0} e^{-\frac{(\theta - x)^2}{2a^2}} d\theta = \sqrt{2\pi}\, a\, \phi\left(\frac{-x}{a}\right)$, where ϕ is the cdf of $Z \sim \mathcal{N}(0,1)$.

$$\Rightarrow \int_{-\infty}^{0} e^{-\frac{\theta^2 - 2x\theta}{2a^2}} d\theta = a\sqrt{2\pi}\, \phi\left(-\frac{x}{a}\right) e^{\frac{x^2}{2a^2}}.$$

Similarly, we have $\int_{0}^{\infty} e^{-\frac{\theta^2 - 2x\theta}{2a^2}} d\theta = a\sqrt{2\pi}\left(1 - \phi\left(-\frac{x}{a^2}\right)\right) e^{\frac{x^2}{2a^2}}$

It follows that:

$$\int_{-\infty}^{\infty} g(\theta)\, d\theta = \int_{-\infty}^{0} g(\theta)\, d\theta + \int_{0}^{\infty} g(\theta)\, d\theta$$

$$\Longrightarrow \int_{-\infty}^{\infty} g(\theta)\,d\theta = \int_{-\infty}^{0} e^{-\frac{\theta^2-2\left(\bar{x}-\frac{\sigma^2}{2\tau}\right)\theta}{2\sigma^2/n}}\,d\theta + \int_{0}^{\infty} e^{-\frac{\theta^2-2\left(\bar{x}+\frac{\sigma^2}{2\tau}\right)\theta}{2\sigma^2/n}}\,d\theta$$

$$= \frac{\sigma\sqrt{2\pi}}{\sqrt{n}}\,\phi\left(-\frac{\bar{x}-\frac{\sigma^2}{\tau n}}{\sigma/\sqrt{n}}\right) e^{\frac{\left(\bar{x}-\frac{\sigma^2}{n\tau}\right)^2}{2\sigma^2/n}} + \frac{\sigma\sqrt{2\pi}}{\sqrt{n}}\left[1-\phi\left(-\frac{\bar{x}+\frac{\sigma^2}{n\tau}}{\sigma/\sqrt{n}}\right)\right] e^{\frac{\left(\bar{x}+\frac{\sigma^2}{n\tau}\right)^2}{2\sigma^2/n}}$$

Hence, the posterior distribution has the following pdf:

$$\pi(\theta \| \underline{x}) = \begin{cases} k\, e^{-\frac{\theta^2-2\left(\bar{x}+\frac{\sigma^2}{n\tau}\right)\theta}{2\sigma^2/n}}, & \text{if } \theta \geq 0 \\ k\, e^{-\frac{\theta^2-2\left(\bar{x}-\frac{\sigma^2}{n\tau}\right)\theta}{2\sigma^2/n}}, & \text{if } \theta < 0, \end{cases}$$

where $k = \left(\int_{-\infty}^{\infty} g(\theta)\,d\theta\right)^{-1}$. Let $m_-(\bar{x},\tau) = \bar{x}+\frac{\sigma^2}{n\tau}$ and $m_+(\bar{x},\tau) = \bar{x}+\frac{\sigma^2}{n\tau}$.

Then $$\frac{1}{k} = \frac{\sigma\sqrt{2\pi}}{\sqrt{n}}\left[\phi\left(-\frac{m_-(\bar{x},\tau)}{\sigma/\sqrt{n}}\right) e^{\frac{m_-^2(\bar{x},\tau)}{2\sigma^2/n}} + \left[1-\phi\left(-\frac{m_+(\bar{x},\tau)}{\sigma/\sqrt{n}}\right)\right] e^{\frac{m_+^2(\bar{x},\tau)}{2\sigma^2/n}}\right].$$

* If $c \geq 0$, then $$P(\theta > c \| \underline{x}) = k\int_{c}^{\infty} e^{-\frac{\theta^2-2m_+(\bar{x},\tau)\theta}{2\sigma^2/n}}\,d\theta$$

$$= k\,\frac{\sigma\sqrt{2\pi}}{\sqrt{n}}\left[1-\phi\left(\frac{c-m_+(\bar{x},\tau)}{\sigma/\sqrt{n}}\right)\right] e^{\frac{m_+^2(\bar{x},\tau)}{2\sigma^2/n}}$$

$$= \frac{\left[1-\phi\left(\frac{c-m_+(\bar{x},\tau)}{\sigma/\sqrt{n}}\right)\right] e^{\frac{m_+^2(\bar{x},\tau)}{2\sigma^2/n}}}{\phi\left[-\frac{m_-^2(\bar{x},\tau)}{\sigma/\sqrt{n}}\right] e^{\frac{m_-^2(\bar{x},\tau)}{2\sigma^2/n}} + \left[1-\phi\left(-\frac{m_+(\bar{x},\tau)}{\sigma/\sqrt{n}}\right)\right] e^{\frac{m_+^2(\bar{x},\tau)}{2\sigma^2/n}}}$$

** If $c < 0$, then $$P(\theta > c \| \underline{x}) = 1 - P(\theta < c \| \underline{x})$$

$$= 1 - k\int_{-\infty}^{c} e^{-\frac{\theta^2-2m_-(\bar{x},\tau)\theta}{2\sigma^2/n}}\,d\theta$$

$$= 1 - k\,\frac{\sigma\sqrt{2\pi}}{\sqrt{n}}\,\phi\left[\frac{c-m_-(\bar{x},\tau)}{\sigma/\sqrt{n}}\right] e^{\frac{m_-^2(\bar{x},\tau)}{2\sigma^2/n}}.$$

$$\rightarrow P(\theta > c \| \underline{x}) = 1 - \frac{\phi\left(\frac{c - m_-(\bar{x},\tau)}{\sigma/\sqrt{n}}\right) e^{\frac{m_-^2(\bar{x},\tau)}{2\sigma^2/n}}}{\phi\left(-\frac{m_-^2(\bar{x},\tau)}{\sigma/\sqrt{n}}\right) e^{\frac{m_-^2(\bar{x},\tau)}{2\sigma^2/n}} + \left[1 - \phi\left(-\frac{m_+(\bar{x},\tau)}{\sigma/\sqrt{n}}\right)\right] e^{\frac{m_+^2(\bar{x},\tau)}{2\sigma^2/n}}}$$

2) Calculate $P(\theta > c \| \underline{x})$, as $\tau \to \infty$:

Notice that if $\tau \to \infty$, $\begin{cases} m_-(\bar{x},\tau) \to \bar{x} \\ m_+(\bar{x},\tau) \to \bar{x}. \end{cases}$

Then, $$\lim_{\tau\to\infty} \frac{1}{k} = \frac{\sigma\sqrt{2\pi}}{\sqrt{n}}\left[\phi\left(-\frac{\bar{x}}{\sigma/\sqrt{n}}\right) e^{\frac{\bar{x}^2}{2\sigma^2/n}} + \left(1 - \phi\left(-\frac{\bar{x}}{\sigma/\sqrt{n}}\right)\right) e^{\frac{\bar{x}^2}{2\sigma^2/n}}\right]$$

$$= \frac{\sigma\sqrt{2\pi}}{\sqrt{n}} e^{\frac{\bar{x}^2}{2\sigma^2/n}}.$$

* If $c \geq 0$, then we have

$$\lim_{\tau\to\infty} P(\theta > c \| \underline{x}) = \frac{1}{\frac{\sigma\sqrt{2\pi}}{\sqrt{n}} e^{\frac{\bar{x}^2}{2\sigma^2/n}}} \frac{\sigma\sqrt{2\pi}}{\sqrt{n}} e^{\frac{\bar{x}^2}{2\sigma^2/n}}\left[1 - \phi\left(\frac{c - \bar{x}}{\sigma/\sqrt{n}}\right)\right] = 1 - \phi\left(\frac{c - \bar{x}}{\sigma/\sqrt{n}}\right).$$

** If $c < 0$, then we have

$$\lim_{\tau\to\infty} P(\theta > c \| \underline{x}) = 1 - \frac{1}{\frac{\sigma\sqrt{2\pi}}{\sqrt{n}} e^{\frac{\bar{x}^2}{2\sigma^2/n}}} \frac{\sigma\sqrt{2\pi}}{\sqrt{n}} e^{\frac{\bar{x}^2}{2\sigma^2/n}} \phi\left(\frac{c - \bar{x}}{\sigma/\sqrt{n}}\right) = 1 - \phi\left(\frac{c - \bar{x}}{\sigma/\sqrt{n}}\right)$$

Therefore, $\lim_{\tau\to\infty} P(\theta > c \| \underline{x}) = 1 - \phi\left(\frac{c - \bar{x}}{\sigma/\sqrt{n}}\right).$

3) - Compare to the p-value associated with the classical hypothesis test:

A classical test reject $H_0 : \theta \leq 0$ if $\frac{\bar{x} - 0}{\sigma/\sqrt{n}}$ is too large.
Therefore, the observed p-value is:

$$p(\underline{x}) = P\left(Z > \frac{\bar{x} - 0}{\sigma/\sqrt{n}}\right) = 1 - \phi\left(\frac{\bar{x}}{\sigma/\sqrt{n}}\right), \text{ where } Z \sim N(0,1).$$

But the result in 1) with $c = 0$ is:

$$\lim_{\tau\to\infty} P(\theta > 0 \| \underline{x}) = 1 - \phi\left(-\frac{\bar{x}}{\sigma/\sqrt{n}}\right) = \phi\left(\frac{\bar{x}}{\sigma/\sqrt{n}}\right).$$

Thus, $\lim_{\tau\to\infty} P(\theta > 0 \| \underline{x}) + p(\underline{x}) = 1.$

Exercise 4:

Let $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(\theta, \sigma^2)$.

$\theta \sim \mathcal{N}(0, \tau^2)$, τ^2 known.

Consider testing (H): $\begin{cases} H_0 : \theta \leq 0 \\ H_1 : \theta > 0. \end{cases}$

1)- Calculate $P(\theta \leq 0 \| \underline{x})$:

Notice that $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is sufficient for θ. Further, $\overline{X} \sim \mathcal{N}\left(\theta, \frac{\sigma^2}{n}\right)$. Therefore, $\pi(\theta \| \underline{x}) = f(\overline{x} \| \theta)\pi(\theta)$

$$\Rightarrow \pi(\theta \| \underline{x}) = \left(\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(\overline{x}-\theta)^2}{2\sigma^2}}\right)\left(\frac{1}{\tau\sqrt{2\pi}} e^{-\frac{\theta^2}{2\tau^2}}\right)$$

$$\propto e^{-\left[\frac{\left(\theta - \frac{\tau^2 \overline{x}}{\tau^2 + \sigma^2/n}\right)^2}{2\left(\frac{\tau^2\sigma^2/n}{\tau^2 + \frac{\sigma^2}{n}}\right)}\right]}$$

Thus, $\theta \| \underline{x} \sim \mathcal{N}\left(\frac{\tau^2}{\tau^2 + \frac{\sigma^2}{n}}\overline{x}, \frac{\tau^2 \frac{\sigma^2}{n}}{\tau^2 + \frac{\sigma^2}{n}}\right)$

Hence, $$P(\theta \leq 0 \| \underline{x}) = \phi\left(\frac{0 - \frac{\tau^2}{\tau^2 + \frac{\sigma^2}{n}}\overline{x}}{\sqrt{\frac{\tau^2\sigma^2/n}{\tau^2 + \frac{\sigma^2}{n}}}}\right) = 1 - \phi\left(\frac{\overline{x}\sqrt{\frac{\tau^2}{\tau^2 + \sigma^2/n}}}{\sigma/\sqrt{n}}\overline{x}\right)$$

2) Since $H_0 : \theta \leq 0$ is rejected if $\frac{\overline{x} - 0}{\sigma/\sqrt{n}}$ is too large, then the observed p-value is:

$$p(\underline{x}) = \sup_{\theta \leq 0} P_\theta(\overline{X} > \overline{x}) = 1 - \phi\left(\frac{\overline{x}}{\sigma/\sqrt{n}}\right)$$

3) If $\sigma^2 = \tau^2 = 1$, then $\frac{\tau^2}{\tau^2 + \frac{\sigma^2}{n}} = \frac{1}{1 + \frac{1}{n}}$ and $\frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{n}}$.

Then, $$P(\theta \leq 0 \| \underline{x}) = 1 - \phi\left(\frac{\overline{x}\sqrt{\frac{1}{1+\frac{1}{n}}}}{1/\sqrt{n}}\right) = 1 - \phi\left(\frac{n}{\sqrt{n+1}}\overline{x}\right)$$

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On the other hand, the observed p-value is

$$p(\underline{x}) = 1 - \phi\left(\frac{\bar{x}}{1/\sqrt{n}}\right) = 1 - \phi(\bar{x}\sqrt{n}).$$

Since $\frac{n}{\sqrt{n+1}} < \sqrt{n}$, then if $\bar{x} > 0$

$$\phi\left(\frac{n}{\sqrt{n+1}}\bar{x}\right) < \phi(\bar{x}\sqrt{n}).$$

Therefore, $P(\theta \leq 0 \| \underline{x}) > p(\underline{x})$ if $\bar{x} > 0$.

4) Notice that if $\tau^2 \to \infty$, then

$$\frac{\tau^2}{\tau^2 + \frac{\sigma^2}{n}} \longrightarrow 1.$$

Therefore, $P(\theta \leq 0 \| \underline{x}) \longrightarrow 1 - \phi\left(\frac{\bar{x}}{\sigma/\sqrt{n}}\right) = p(\underline{x})$

Thus $P(\theta \leq 0 \| \underline{x}) \underset{\tau \to \infty}{\longrightarrow} p(\underline{x})$.