Exercice 1:

$X_1, \cdots, X_n \overset{iid}{\sim} \mathcal{B}(\mu,1) \Rightarrow f_X(x\|\mu) = \dfrac{x^{\mu-1}}{B(\mu,1)}$, $x \in (0,1)$, $\mu > 0$.

$Y_1, \cdots, Y_m \overset{iid}{\sim} \mathcal{B}(\theta,1) \Rightarrow f_Y(y\|\theta) = \dfrac{y^{\theta-1}}{B(\theta,1)}$, $y \in (0,1)$, $\theta > 0$.

X's are independent of Y's.

1)- Find an LRT for (H): $\begin{cases} H_0 : \theta = \mu \\ H_1 : \theta \neq \mu. \end{cases}$

- By definition, $L(\mu,\theta \| \underline{x}, \underline{y}) = f_{(X,Y)}(\underline{x},\underline{y}\|\mu,\theta)$

$$= f_X(\underline{x}\|\mu) f_Y(\underline{y}\|\theta) \text{, since } X \perp\!\!\!\perp Y$$

$$= \left(\prod_{i=1}^{n} f_X(x_i\|\mu)\right)\left(\prod_{j=1}^{m} f_Y(y_i\|\theta)\right)$$

$$= \left(\prod_{i=1}^{n} \frac{\Gamma(\mu+1)}{\Gamma(\mu)\Gamma(1)} x_i^{\mu-1}\right)\left(\prod_{j=1}^{m} \frac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma(1)} y_j^{\theta-1}\right)$$

$$= \mu^n \theta^m \left(\prod_{i=1}^{n} x_i\right)^{\mu-1} \left(\prod_{j=1}^{m} y_j\right)^{\theta-1} \text{, since } \forall x>0,\ \Gamma(x+1) = x\Gamma(x).$$

So, $\ell(\mu,\theta\|\underline{x},\underline{y}) = \ln L(\mu,\theta\|\underline{x},\underline{y})$

$$\ell(\mu,\theta\|\underline{x},\underline{y}) = n\ln\mu + m\ln\theta + (\mu-1)\sum_{i=1}^{n}\ln x_i + (\theta-1)\sum_{j=1}^{m}\ln y_j \quad (*)$$

$$\Rightarrow \begin{cases} \dfrac{\partial \ell}{\partial \mu} = \dfrac{n}{\mu} + \sum_{i=1}^{n} \ln x_i = 0 \\ \dfrac{\partial \ell}{\partial \theta} = \dfrac{m}{\theta} + \sum_{j=1}^{m} \ln y_j = 0 \end{cases} \Rightarrow \begin{cases} \hat{\mu} = -\dfrac{n}{\sum_{i=1}^{n} \ln x_i} \\ \hat{\theta} = -\dfrac{m}{\sum_{j=1}^{m} \ln y_j} \end{cases}$$

So, $(\hat{\mu},\hat{\theta})$ is a candidate for a MLE of (μ,θ).

Further, we have $\begin{cases} \left.\dfrac{\partial^2 l}{\partial \mu^2}\right|_{\mu=\hat{\mu}} = -\dfrac{n}{\hat{\mu}^2} < 0 \; ; \; \left.\dfrac{\partial^2 l}{\partial \theta^2}\right|_{\theta=\hat{\theta}} = -\dfrac{m}{\hat{\theta}^2} < 0 \\ Hl(\hat{\mu},\hat{\theta} \| \underline{x}, \underline{y}) = \begin{vmatrix} -\frac{n}{\hat{\mu}^2} & 0 \\ 0 & -\frac{m}{\hat{\theta}^2} \end{vmatrix} = \dfrac{nm}{(\hat{\mu}\hat{\theta})^2} > 0 \end{cases}$

Therefore, $\left(-\dfrac{n}{\sum_{i=1}^{n} \ln X_i}, -\dfrac{m}{\sum_{j=1}^{m} \ln Y_j} \right)$ is a MLE for (μ, θ).

- Now, let $(\hat{\mu}_0, \hat{\theta}_0)$ denote the restricted MLE of (μ,θ). Under $H_0 : \theta = \mu$, $\hat{\theta}_0 = \hat{\mu}_0$ and $(*)$ becomes:

$$l(\theta \| \underline{x}, \underline{y}) = (n+m)\ln\theta + (\theta-1)\left(\sum_{i=1}^{n} \ln x_i + \sum_{j=1}^{m} \ln y_j \right)$$

$$\Rightarrow \frac{\partial l}{\partial \theta} = \frac{n+m}{\theta} + \sum_{i=1}^{n} \ln x_i + \sum_{j=1}^{m} \ln y_j$$

Setting $\frac{\partial l}{\partial \theta} = 0$ yields $\hat{\theta}_0 = -\dfrac{(n+m)}{\sum_{i=1}^{n} \ln x_i + \sum_{j=1}^{m} \ln y_j}$

Notice that $\left.\dfrac{\partial^2 l}{\partial \theta^2}\right|_{\theta=\hat{\theta}_0} < 0$.

So, $\hat{\theta}_0 = -\dfrac{(n+m)}{\sum_{i=1}^{n} \ln X_i + \sum_{j=1}^{m} \ln Y_j}$ is the MLE for (μ,θ) under $H_0 : \theta = \mu$.

Therefore, the LRT statistic of (H) is:

$$\lambda(\underline{x}, \underline{y}) = \frac{L(\hat{\theta}_0, \hat{\theta}_0 \| \underline{x}, \underline{y})}{L(\hat{\mu}, \hat{\theta} \| \underline{x}, \underline{y})} = \frac{\hat{\theta}_0^{\,n+m} \left(\prod_{i=1}^{n} x_i \prod_{j=1}^{m} y_j \right)^{\hat{\theta}_0 - 1}}{\hat{\mu}^n \hat{\theta}^m \left(\prod_{i=1}^{n} x_i \right)^{\hat{\mu}-1} \left(\prod_{j=1}^{m} y_j \right)^{\hat{\theta}-1}}$$

$$= \left(\frac{\hat{\theta}_0}{\hat{\mu}} \right)^n \left(\frac{\hat{\theta}_0}{\hat{\theta}} \right)^m \left(\prod_{i=1}^{n} x_i \right)^{\hat{\theta}_0 - \hat{\mu}} \left(\prod_{j=1}^{m} y_j \right)^{\hat{\theta}_0 - \hat{\theta}}.$$

$$= \left(\frac{\hat{\theta}_0}{\hat{\mu}} \right)^n \left(\frac{\hat{\theta}_0}{\hat{\theta}} \right)^m.$$

In fact, $\left(\prod_{i=1}^{n} x_i\right)^{\hat{\theta}_0-\hat{\mu}}\left(\prod_{j=1}^{m} y_j\right)^{\hat{\theta}_0-\hat{\theta}} = e^{(\hat{\theta}_0-\hat{\mu})\sum_{i=1}^{n}\ln x_i+(\hat{\theta}_0-\hat{\theta})\sum_{j=1}^{m}\ln y_j}$

$$= e^{\hat{\theta}_0\left(\sum_{i=1}^{n}\ln x_i+\sum_{j=1}^{m}\ln y_j\right)-\hat{\mu}\sum_{i=1}^{n}\ln x_i-\hat{\theta}\sum_{j=1}^{m}\ln y_j}$$

$$= e^{\hat{\theta}_0\left(-\frac{n+m}{\hat{\theta}_0}\right)-\hat{\mu}\left(-\frac{n}{\hat{\mu}}\right)-\hat{\theta}\left(-\frac{m}{\hat{\theta}}\right)}$$

$$= e^{-(n+m)+n+m} = e^0 = 1.$$

Thus, a size α LRT test for $(H): \begin{cases} H_0 : \theta = \mu \\ H_1 : \theta \neq \mu \end{cases}$ is

$$\phi(\underline{x},\underline{y}) = \begin{cases} 1, & \left(\frac{\hat{\theta}_0}{\hat{\mu}}\right)^n\left(\frac{\hat{\theta}_0}{\hat{\theta}}\right)^m < k \\ 0, & \text{otherwise.} \end{cases}$$

2)- Show that the test in part (1) can be based on the test statistic : $T = \dfrac{\sum_{i=1}^{n}\ln X_i}{\sum_{i=1}^{n}\ln X_i+\sum_{j=1}^{m}\ln Y_j}$

We have $\lambda(\underline{x},\underline{y}) = \left(\frac{\hat{\theta}_0}{\hat{\mu}}\right)^n\left(\frac{\hat{\theta}_0}{\hat{\theta}}\right)^m$

$$= \left[\frac{(n+m)\sum_{i=1}^{n}\ln x_i}{n\left(\sum_{i=1}^{n}\ln x_i+\sum_{j=1}^{m}\ln y_j\right)}\right]^n\left[\frac{(n+m)\sum_{j=1}^{m}\ln y_j}{m\left(\sum_{i=1}^{n}\ln x_i+\sum_{j=1}^{m}\ln y_j\right)}\right]^m$$

$$= \left[\frac{n+m}{n}t\right]^n\left[\frac{n+m}{m}(1-t)\right]^m, \text{ as } \frac{y}{x+y} = 1-\frac{x}{x+y}.$$

$$= \left(\frac{n+m}{n}\right)^n\left(\frac{n+m}{m}\right)^m t^n(1-t)^m = g(t)$$

Therefore, the LRT test ϕ can be based on T.

Thus, $\lambda(\underline{x},\underline{y}) < k \iff \left(\frac{n+m}{n}\right)^n \left(\frac{n+m}{m}\right)^m t^n (1-t)^m < k$

$$\iff t^n(1-t)^m < c$$

$$\iff t < c_1 \text{ or } t > c_2 \quad (c_1 < c_2).$$

In fact, setting $h(t) = t^n(1-t)^m$, $0<t<1$, yields

$$h'(t) = t^{n-1}(1-t)^{m-1}\left[n-(n+m)t\right].$$

So, $h'(t) = 0 \Rightarrow t_0 = \frac{n}{n+m}$.

t	0		t_0		1
$h'(t)$		$+$	0	$-$	
$h(t)$	0	$\nearrow$	1	$\searrow$	0

Hence, an LRT for (H) is $\phi(t) = \begin{cases} 1, & t<c_1 \text{ or } t>c_2 \\ 0, & \text{otherwise,} \end{cases}$

where the scalars c_1 and c_2 satisfy: $P_\theta(T<c_1) + P_\theta(T>c_2) = \alpha$.

3)- Find the distribution of T when H_0 is true, and then show how to get a test of size $\alpha = 0{,}10$.

Under $H_0: \theta = \mu$, $X_1, \dots, X_n, Y_1, \dots, Y_m \overset{iid}{\sim} \mathcal{B}(\theta, 1)$.

Further, notice that $T = \dfrac{-\sum_{i=1}^{n} \ln X_i}{-\sum_{i=1}^{n} \ln X_i - \sum_{j=1}^{m} \ln Y_j}$

Set $Z_i = -\ln X_i$. Let $z = -\ln x \Rightarrow x = e^{-z} = \omega(z)$

The change of variable theorem yields that

$$g_{Z_i}(z) = |\omega'(z)|\, f_X(\omega(z)) = \theta e^{-z} e^{-(\theta-1)z} = \theta e^{-\theta z}, \quad z>0.$$

Therefore, $-\ln X_i \sim \mathcal{E}xp(\theta)$. Likewise, $-\ln Y_i \sim \mathcal{E}xp(\theta)$.

Thus, $-\sum_{i=1}^{n} \ln X_i \sim \Gamma(n, \theta)$ and $-\sum_{j=1}^{m} \ln Y_j \sim \Gamma(m, \theta)$.

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Hence, $T \sim \mathcal{B}(n, m)$, as $-\sum_{i=1}^{n} \ln X_i \perp\!\!\!\perp -\sum_{j=1}^{m} \ln Y_j$.

As a result, a test ϕ_1 of size $\alpha = 0{,}10$ can be obtained as follows:

$$\phi_1(t) = \begin{cases} 1, & t < c_1 \text{ or } t > c_2 \\ 0, & \text{otherwise,} \end{cases} \qquad (c_1 < c_2)$$

where the scalars c_1 and c_2 are such that,

$$P(T < c_1) + P(T > c_2) = 0{,}10, \text{ with } T \sim \mathcal{B}(n, m).$$

Exercise 2:

Let $(X_1, \cdots, X_n)$ be a random sample from a population with pdf:

$$f(x\|\theta,\lambda) = \frac{1}{\lambda} e^{-\frac{(x-\theta)}{\lambda}} \mathbb{1}_{[\theta,\infty)}(x), \quad \theta\in\mathbb{R},\ \lambda>0.$$

1)- Find the LRT of (H): $\begin{cases} H_0 : \theta \leq 0 \\ H_1 : \theta > 0. \end{cases}$

- By definition, the LRT of (H) is $\lambda(\underline{x}) = \dfrac{\sup_{\theta\leq 0} L(\theta,\lambda\|\underline{x})}{\sup_{\theta\in\mathbb{R}} L(\theta,\lambda\|\underline{x})}$,

where $L(\theta,\lambda\|\underline{x}) = \prod_{i=1}^{n} f(x_i\|\theta,\lambda)$

$$= \frac{1}{\lambda^n} e^{-\frac{1}{\lambda}\sum_{i=1}^{n}(x_i-\theta)} \mathbb{1}_{(-\infty, x_{(1)}]}(\theta), \text{ with } x_{(1)} = \min_{1\leq i\leq n} x_i.$$

Notice that $\forall \lambda>0$, $\theta \mapsto L(\theta,\lambda\|\underline{x})$ is nondecreasing.

So, its maximum is attained at $\theta = x_{(1)}$.

Thus, the MLE of θ is $\hat{\theta} = \min_{1\leq i\leq n} X_i = X_{(1)}$.

Moreover, $l(x_{(1)},\lambda) = \ln L(x_{(1)},\lambda\|\underline{x})$.

$$l(x_{(1)},\lambda) = -n\ln\lambda - \frac{1}{\lambda}\sum_{i=1}^{n}(x_i - x_{(1)}).$$

$$\Rightarrow \frac{dl}{d\lambda} = -\frac{n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n}(x_i - x_{(1)}) = 0 \Rightarrow \hat{\lambda} = \bar{x} - x_{(1)}$$

is a candidate for MLE of λ. Furthermore,

λ	0	$\bar{x}-x_{(1)}$	$+\infty$
$\frac{dl}{d\lambda}$	$+$	0	$-$

$$\left(\left.\frac{d^2 l}{d\lambda^2}\right|_{\lambda=\bar{x}-x_{(1)}} = -\frac{n}{(\bar{x}-x_{(1)})^2} < 0 \right).$$

So, the MLE for λ is $\hat{\lambda} = \bar{X} - X_{(1)}$.

- Under $H_0 : \theta \leq 0$, $\hat{\theta}_0 = \begin{cases} x_{(1)}, & x_{(1)} \leq 0 \\ 0, & x_{(1)} > 0 \end{cases}$ and $\hat{\lambda}_0 = \bar{x} - \hat{\theta}_0$.

Therefore, the LRT statistic of (H) is $\lambda(\underline{x}) = \begin{cases} 1 & , x_{(1)} \leq 0 \\ \left(1-\frac{x_{(1)}}{\bar{x}}\right)^n & , x_{(1)} > 0. \end{cases}$

In fact, if $x_{(1)} > 0$, then $\lambda(\underline{x}) = \frac{L(\bar{x}, 0 \| \underline{x})}{L(x_{(1)}, \bar{x} - x_{(1)} \| \underline{x})}$.

$$\Rightarrow \lambda(\underline{x}) = \frac{(1/\bar{x})^n e^{-\frac{n\bar{x}}{\bar{x}}}}{\left(\frac{1}{\bar{x}-x_{(1)}}\right)^n e^{-n\frac{(\bar{x}-x_{(1)})}{\bar{x}-x_{(1)}}}} = \left(\frac{\bar{x}-x_{(1)}}{\bar{x}}\right)^n = \left(1-\frac{x_{(1)}}{\bar{x}}\right)^n.$$

Thus, a size α LRT test ϕ rejects $H_0: \theta \leq 0$ if $\lambda(\underline{x}) \leq k$, $k \in (0,1)$. This is equivalent to:

$$\left(1-\frac{x_{(1)}}{\bar{x}}\right)^n \leq k \Leftrightarrow 1 - \frac{x_{(1)}}{\bar{x}} \leq c$$
$$\Leftrightarrow \frac{x_{(1)}}{\bar{x}} \geq c_1.$$

Hence, a nontrivial size α LRT for (H) is

$$\phi(\underline{x}) = \begin{cases} 1 & , \frac{x_{(1)}}{\bar{x}} > c_1 \\ 0 & , \text{o.w.} \end{cases}$$

2)- Let $(X_1, \ldots, X_n)$ be a random sample such that:

$$f(x \| \gamma, \beta) = \frac{\gamma}{\beta} x^{\gamma-1} e^{-\frac{1}{\beta}x^\gamma}, \quad x > 0, \ \gamma, \beta > 0.$$

<u>Find the LRT of (H)</u>: $\begin{cases} H_0: \gamma = 1 \\ H_1: \gamma \neq 1. \end{cases}$

By definition, $L(\gamma, \beta \| \underline{x}) = \prod_{i=1}^n f(x_i \| \gamma, \beta) = \left(\frac{\gamma}{\beta}\right)^n \left(\prod_{i=1}^n x_i\right)^{\gamma-1} e^{-\frac{1}{\beta}\sum_{i=1}^n x_i^\gamma}$

$$\Rightarrow \ell(\gamma, \beta) = \ln L(\gamma, \beta \| \underline{x}) = n\ln\gamma - n\ln\beta + (\gamma-1)\sum_{i=1}^n \ln x_i - \frac{1}{\beta}\sum_{i=1}^n x_i^\gamma$$

$$\Rightarrow \begin{cases} \frac{\partial \ell(\gamma, \beta)}{\partial \beta} = -\frac{n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^n x_i^\gamma = 0 & (i) \\ \frac{\partial \ell(\gamma, \beta)}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^n \ln x_i - \frac{1}{\beta}\sum_{i=1}^n x_i^\gamma \ln x_i = 0 & (ii) \end{cases}$$

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(i) $\Rightarrow \beta = \frac{1}{n}\sum_{i=1}^{n} x_i^{\gamma}$. Plugging this in (ii) yields:

$$\frac{n}{\gamma} + \sum_{i=1}^{n} \ln x_i = \frac{n\sum_{i=1}^{n} x_i^{\gamma}\ln x_i}{\sum_{i=1}^{n} x_i^{\gamma}}$$

$$\Rightarrow \frac{1}{\gamma} + \frac{1}{n}\sum_{i=1}^{n}\ln x_i = \sum_{i=1}^{n}\frac{x_i^{\gamma}}{\sum_{i=1}^{n} x_i^{\gamma}}\ln x_i .$$

$$\overset{(iii)}{\Rightarrow} \frac{1}{\gamma} + \frac{1}{n}\sum_{i=1}^{n}\ln x_i = \sum_{i=1}^{n}\omega_i \ln x_i \ , \text{ with } \omega_i = \frac{x_i^{\gamma}}{\sum_{i=1}^{n} x_i^{\gamma}}, \forall i=\overline{1,n}.$$

Notice that as x_i gets large so does ω_i.
So, the RHS of iii) is nondecreasing in γ
while its LHS is nonincreasing in γ,
Further as $\gamma \to 0$, $\omega_i \to \frac{1}{n}$. Thus, iii) has
a unique solution. W.L.O.G, the MLEs are
the solution of iii), ie $\hat{\gamma}$ and $\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} X_i^{\hat{\gamma}}$.

- Under $H_0: \gamma = 1$, the restricted MLEs are $\hat{\gamma}_0 = 1$ and $\hat{\beta}_0 = \bar{X}$.
- It follows that the LRT statistic for (H) is

$$\lambda(\underline{x}) = \frac{L(1,\bar{x}\|\underline{x})}{L(\hat{\gamma},\hat{\beta}\|\underline{x})} = \frac{(\bar{x})^{-n}e^{-n}}{\left(\frac{\hat{\gamma}}{\hat{\beta}}\right)^{n}\left(\prod_{i=1}^{n} x_i\right)^{\hat{\gamma}-1} e^{-\frac{n\hat{\beta}}{\hat{\beta}}}} = \frac{1}{\bar{x}^{n}\hat{\gamma}^{n}\left(\prod_{i=1}^{n} x_i\right)^{\hat{\gamma}-1}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\hat{\gamma}}\right)^{n}}.$$

Thus, an LRT ϕ rejects $H_0: \gamma = 1$ if $\lambda(\underline{x}) < k$, $(k \in (0,1))$.

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Exercise 3:

Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Pareto}(\theta, \nu)$

$$\Rightarrow f(x \| \theta, \nu) = \frac{\theta \nu^\theta}{x^{\theta+1}} \mathbb{1}_{[\nu, +\infty)}(x), \quad \theta > 0, \nu > 0.$$

1)- <u>Find the MLEs for θ and ν:</u>

By definition, $L(\theta, \nu \| \underline{x}) = \prod_{i=1}^n f(x_i \| \theta, \nu) = \left(\prod_{i=1}^n \frac{\theta \nu^\theta}{x_i^{\theta+1}} \right) \left(\prod_{i=1}^n \mathbb{1}_{(0, x_i]}(\nu) \right)$

$$= \frac{\theta^n \nu^{n\theta}}{\left(\prod_{i=1}^n x_i \right)^{\theta+1}} \mathbb{1}_{(0, x_{(1)}]}(\nu), \quad x_{(1)} = \min_{1 \le i \le n} x_i.$$

Notice that for fixed $\theta > 0$, $\nu \mapsto L(\theta, \nu \| \underline{x})$ is nondecreasing on $(0, x_{(1)}]$. So, $\nu \mapsto L(\theta, \nu \| \underline{x})$ attains its maximum at $\nu = x_{(1)}$. Thus, the MLE for ν is $\hat{\nu} = X_{(1)}$.

Further, $l(\theta, x_{(1)} \| \underline{x}) = \ln L(\theta, x_{(1)} \| \underline{x})$

$$l(\theta, x_{(1)} \| x) = n \ln \theta + n\theta \ln x_{(1)} - (\theta+1) \sum_{i=1}^n \ln x_i$$

$$\Rightarrow \frac{dl}{d\theta} = \frac{n}{\theta} + n \ln x_{(1)} - \sum_{i=1}^n \ln x_i$$

Now $\frac{dl}{d\theta} = 0 \Leftrightarrow \ln\left(\prod_{i=1}^n x_i \right) - \ln x_{(1)}^n = \frac{n}{\theta}$

$$\rightarrow \hat{\theta} = \frac{n}{\ln\left(\prod_{i=1}^n \frac{x_i}{x_{(1)}} \right)}$$

Since $\left. \frac{d^2 l(\theta, x_{(1)} \| \underline{x})}{d\theta^2} \right|_{\theta = \hat{\theta}} = -\frac{n}{\hat{\theta}^2} < 0$, then $\hat{\theta} = \frac{n}{\ln\left(\prod_{i=1}^n \frac{X_i}{X_{(1)}} \right)}$ is the MLE for θ. Set $T = \ln\left(\prod_{i=1}^n \frac{X_i}{X_{(1)}} \right)$.

d)- <u>Show that the LRT of</u> $(H)\begin{cases} H_0: \theta=1, & \nu\text{-unknown.} \\ H_1: \theta\neq 1, & \nu\text{-unknown.}\end{cases}$

<u>has a critical region of the form</u> $\{\underline{x}: T(\underline{x})\leq c_1 \cup T(\underline{x})\geq c_2\}$

By definition, $\lambda(\underline{x}) = \dfrac{L(\hat{\theta}_0,\hat{\nu}_0 \| \underline{x})}{L(\hat{\theta},\hat{\nu}\|\underline{x})}$, where $(\hat{\theta}_0,\hat{\nu}_0)$ is the MLE for (θ,ν) under H_0.

Under $H_0:\theta=1$, $\hat{\nu}_0 = x_{(1)}$ and $\hat{\theta}_0=1$. Recalling that

$t = \ln\left(\prod_{i=1}^{n}\dfrac{x_i}{x_{(1)}}\right) \Rightarrow \prod_{i=1}^{n} x_i = x_{(1)}^n e^t$. Therefore, we have

$$\lambda(\underline{x}) = \frac{L(1, x_{(1)}\|\underline{x})}{L(\hat{\theta},\hat{\nu}\|\underline{x})} = \frac{x_{(1)}^n}{\left(\prod_{i=1}^{n}x_i\right)^2} \times \frac{t^n\left(\prod_{i=1}^{n}x_i\right)^{\frac{n}{t}+1}}{n^n x_{(1)}^{\frac{n^2}{t}}}$$

$$= \left(\frac{t}{n}\right)^n \left[x_{(1)}\right]^{n-\frac{n^2}{t}} \left(\prod_{i=1}^{n}x_i\right)^{\frac{n}{t}-1} = \left(\frac{t}{n}\right)^n \left[x_{(1)}\right]^{n-\frac{n^2}{t}+\frac{n^2}{t}-n} \left(e^t\right)^{\frac{n}{t}-1}$$

$$= \left(\frac{t}{n}\right)^n e^{n-t} = h_n(t).$$

Hence, a size α LRT ϕ rejects $H_0:\theta=1$ if $h_n(t)<k$, $k\in(0,1)$.

But $h_n'(t) = \left(\dfrac{t}{n}\right)^{n-1} e^{n-t}\left(1-\dfrac{t}{n}\right) = 0 \rightarrow t_0 = n$.

t	0		t_0		$+\infty$
$h_n'(t)$		$+$	0	$-$	
$h_n(t)$	0	$\nearrow$	1	$\searrow$	0

Thus $h_n(t)<k \Leftrightarrow t<c_1$ or $t>c_2$, $(c_1<c_2)$.

Hence an LRT for (H) is $\phi(t) = \begin{cases} 1, & t<c_1 \text{ or } t>c_2 \\ 0, & \text{otherwise,}\end{cases}$

with $P_{\theta=1}(T<c_1) + P_{\theta=1}(T>c_2) = \alpha$.

3)- <u>Show that under H_0, $2T \sim \chi^2$ distribution and find the degrees of freedom:</u>

- In what follows, let $\forall \nu > 0$, $\begin{cases} z_i = \ln\left(\frac{X_i}{\nu}\right), & i = \overline{1,n}. \\ z_{(1)} = \ln\left(\frac{X_{(1)}}{\nu}\right). \end{cases}$

$\forall t \in \mathbb{R}$, we have $P(T \leq t) = P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{X_{(1)}}\right) \leq t\right)$

$$= P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{\nu}\right) - n\ln\left(\frac{X_{(1)}}{\nu}\right) \leq t\right)$$

$$= P\left(\sum_{i=1}^{n} z_i - n z_{(1)} \leq t\right).$$

Therefore, $T \stackrel{d}{=} \sum_{i=1}^{n} z_i - n z_{(1)} \Rightarrow \sum_{i=1}^{n} z_i \stackrel{d}{=} T + n z_{(1)}$. $(*)$

- Under $H_0: \theta = 1$, $f(x \| \theta, \nu) = \frac{\nu}{x^2} \mathbb{1}_{[\nu, \infty)}(x)$, $\nu > 0$.

Let $z = \ln\frac{x}{\nu} \Rightarrow x = \nu e^{z} = \omega(z)$. The change of variable theorem yields that $g_{z_i}(z) = f_X(\omega(z)) |\omega'(z)|$.

So, $g_{z_i}(z) = \frac{\nu}{\nu^2 e^{2z}} (\nu e^{z}) = e^{-z}$, $z > 0$.

Thus, $z_1, \ldots, z_n \stackrel{iid}{\sim} Exp(1)$. As a result,

$$\begin{cases} \sum_{i=1}^{n} z_i \sim \Gamma(n,1) \\ z_{(1)} \sim Exp(n) \end{cases} \Rightarrow \begin{cases} \sum_{i=1}^{n} z_i \sim \Gamma(n,1) \\ n z_{(1)} \sim \Gamma(1,1). \end{cases}$$

Since $T \perp\!\!\!\perp n z_{(1)}$, then $(*)$ implies that

$$M_{\sum_{i=1}^{n} z_i}(t) = M_T(t)\, M_{n z_{(1)}}(t) \Leftrightarrow \frac{1}{(1-t)^n} = M_T(t) \left(\frac{1}{1-t}\right),\ t < 1$$

$\Rightarrow M_T(t) = \frac{1}{(1-t)^{n-1}}$, $t < 1$ $\Rightarrow T \sim \Gamma(n-1, 1)$, by uniqueness of mgf.

$$\Rightarrow 2T \sim \Gamma\left(n-1, \tfrac{1}{2}\right) \equiv \chi^2_{2(n-1)}.$$

Exercice 4:

Let $X, Y \overset{iid}{\sim} \mathcal{U}(\theta, \theta+1)$. Consider $(H): \begin{cases} H_0: \theta = 0 \\ H_1: \theta > 0. \end{cases}$

and $\phi_1(X) = 1 \Leftrightarrow X > 0{,}95$

$\phi_2(X,Y) = 1 \Leftrightarrow X+Y > k$.

1)- Find k so that ϕ_2 has the same size as ϕ_1.

Under $H_0: \theta=0$, $X, Y \overset{iid}{\sim} \mathcal{U}(0,1)$.

Since $\underset{\theta=0}{E}[\phi_1(X)] = \underset{\theta=0}{E}[\phi_2(X,Y)]$, then

$$P(X>0{,}95) = P(X+Y>k)$$

$$1 - 0{,}95 = \int_{k-1}^{1}\int_{k-x}^{1} dy\,dx \quad , \quad \text{as } P(U \le u) = u, \text{ if } U \sim \mathcal{U}(0,1).$$

$$0{,}05 = \frac{1}{2}(2-k)^2$$

Therefore, $k = 2 - \sqrt{0{,}1} = 1{,}684$.

2)- Calculate the power of ϕ_1 and ϕ_2. Graph a well labelled graph of each power function

By definition, $\beta_{\phi_1}(\theta) = P_\theta(X > 0{,}95)$, with $X \sim \mathcal{U}(\theta, \theta+1)$.

We consider the following cases:

i) If $0{,}95 < \theta$, then $\beta_{\phi_1}(\theta) = \int_{0{,}95}^{\theta} f(x)dx + \int_{\theta}^{\theta+1} f(x)dx = \int_{\theta}^{\theta+1} dx = 1$

ii) If $\theta \le 0{,}95 < \theta+1 \Leftrightarrow -0{,}05 < \theta \le 0{,}95$, then $\beta_{\phi_1}(\theta) = \int_{0{,}95}^{\theta+1} dx = \theta + 0{,}05$.

iii) If $0{,}95 \geq \theta+1 \Leftrightarrow \theta \le -0{,}05$, then $\beta_{\phi_1}(\theta) = 0$.

All in all, $\beta_{\phi_1}(\theta) = \begin{cases} 0 & , \ \theta \le -0{,}05 \\ \theta + 0{,}05, & -0{,}05 < \theta \le 0{,}95 \\ 1 \ , & \theta > 0{,}95. \end{cases}$

Question 6. [2 marks]
Use implicit differentiation to find the slope of the tangent line to the curve

$$xe^x + x^2y^2 + y = 1$$

at point $(0, 1)$.

Let $Z = X+Y$. So, $\beta_{\phi_2}(\theta) = P_\theta(Z > k) = \int_k^{2\theta+2} f(z)\,dz$

By convolution theorem, $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$.

Since $X, Y \overset{iid}{\sim} U(\theta, \theta+1)$, then

$$f_X(x) f_Y(z-x) = \begin{cases} 1, & \theta < x < \theta+1 \text{ and } \theta < z-x < \theta+1 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we consider the following cases:

i) $2\theta \le z < 2\theta+1 \Rightarrow f_Z(z) = \int_\theta^{z-\theta} dx = z - 2\theta$

ii) $2\theta+1 \le z < 2\theta+2 \Rightarrow f_Z(z) = \int_{z-(\theta+1)}^{\theta+1} dx = 2(\theta+1) - z$

Thus, $f_Z(z) = \begin{cases} z - 2\theta, & 2\theta \le z < 2\theta+1 \\ 2(\theta+1) - z, & 2\theta+1 \le z < 2\theta+2 \\ 0, & \text{otherwise.} \end{cases}$

Now, to determine $\beta_{\phi_2}(\theta)$, we need to consider the following cases:

i) If $k < 2\theta \Leftrightarrow \theta > \frac{k}{2}$, then

$$\beta_{\phi_2}(\theta) = \int_k^{2\theta} f(z)\,dz + \int_{2\theta}^{2\theta+1} f(z)\,dz + \int_{2\theta+1}^{2\theta+2} f(z)\,dz$$

$$\Rightarrow \beta_{\phi_2}(\theta) = \int_{2\theta}^{2\theta+1} (z-2\theta)\,dz + \int_{2\theta+1}^{2\theta+2} (2\theta+2-z)\,dz$$

$$= \left[\frac{(z-2\theta)^2}{2}\right]_{2\theta}^{2\theta+1} - \left[\frac{(2\theta+2-z)^2}{2}\right]_{2\theta+1}^{2\theta+2} = \frac{1}{2}+\frac{1}{2} = 1.$$

ii) If $2\theta \le k < 2\theta+1 \Leftrightarrow \frac{k-1}{2} < \theta \le \frac{k}{2}$, then

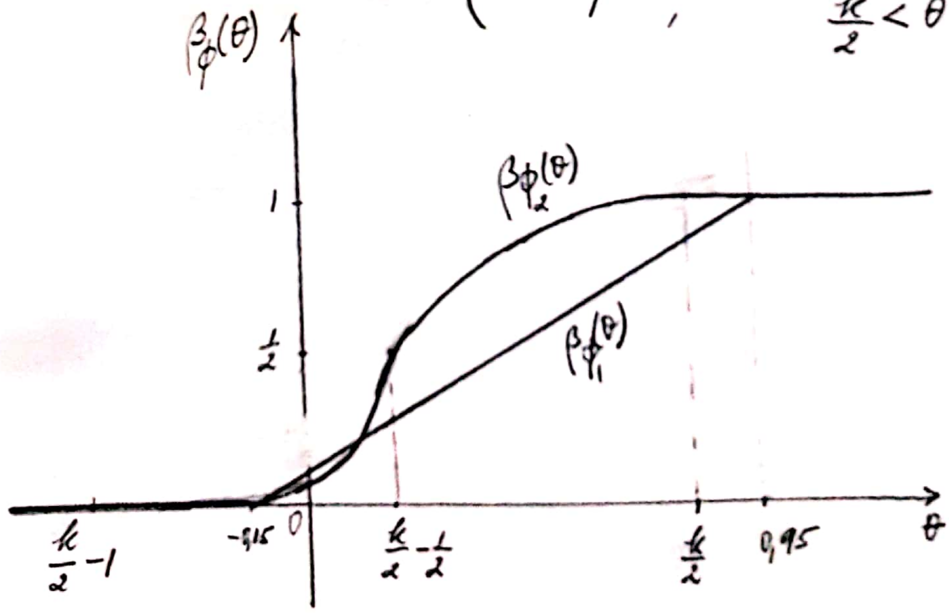
$$\beta_{\phi_2}(\theta) = \int_{k}^{2\theta+1} (z-2\theta)\,dz + \int_{2\theta+1}^{2\theta+2} (2\theta+2-z)\,dz = \left[\frac{(z-2\theta)^2}{2}\right]_{k}^{2\theta+1} + \frac{1}{2} = 1 - \frac{(k-2\theta)^2}{2}.$$

iii) If $2\theta+1 \le k < 2\theta+2 \Leftrightarrow \frac{k}{2}-1 < \theta \le \frac{k-1}{2}$

$$\beta_{\phi_2}(\theta) = \int_{k}^{2\theta+2} (2\theta+2-z)\,dz = \left[-\frac{(2\theta+2-z)^2}{2}\right]_{k}^{2\theta+2} = \frac{(2\theta+2)-k}{2}$$

iv) If $k \geq 2\theta+2 \Leftrightarrow \theta \le \frac{k}{2}-1$, then $\beta_{\phi_2}(\theta) = 0$.

In summary, $$\beta_{\phi_2}(\theta) = \begin{cases} 0 & , \ \theta \le \frac{k}{2}-1 \\ \frac{(2\theta+2-k)^2}{2}, & \frac{k}{2}-1 < \theta \le \frac{k-1}{2} \\ 1-\frac{(k-2\theta)^2}{2}, & \frac{k-1}{2} < \theta < \frac{k}{2} \\ 1 & , \ \frac{k}{2} < \theta \end{cases}$$



Scanné avec CamScanner

Prove or disprove ϕ_2 is more powerful than ϕ_1

i) It follows from the graph that ϕ_2 is more powerful than ϕ_1 except for a small set of alternatives around $\theta=0$. In other words, ϕ_1 is more powerful than ϕ_2 for values of θ near 0 but ϕ_2 is more powerful than ϕ_1 for larger values of θ. Moreover, ϕ_2 is not uniformly more powerful than ϕ_1.

4)- Show how to get a test that has the same size but is more powerful than ϕ_2:

Notice that if $X>c$, $(1<c<k)$ then $H_0: \theta=0$ should be rejected. In fact, $P_{\theta=0}(X<c)=1$. As such, we consider the following test ϕ_3 defined by:

$$\phi_3(x,y)=\begin{cases} 1, & x+y>k \text{ or } x>c \\ 0, & \text{otherwise} \end{cases}$$

Since $\beta_{\phi_3}(0)=P_{\theta=0}(X+Y>k)=\beta_{\phi_2}(0)$, as $P_{\theta=0}(X<c)=1$.

then ϕ_3 has the same size as ϕ_2.

Furthermore, since $\phi_3 \geq \phi_2$, then $\beta_{\phi_3}(\theta) \geq \beta_{\phi_2}(\theta)$. Note that

$\beta_{\phi_3}(\theta) > \beta_{\phi_2}(\theta)$, for $0<\theta<k-c$, as $P_\theta(X+Y<k, Y>c)>0$.

Thus, ϕ_3 is more powerful than ϕ_2.