Mathematical Statistics II

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Outline

Likelihood Ratio Tests

- Let X be a random variable such that $X \sim f(x\|\theta)$, where $\theta \in \Theta$ is a population parameter and $\Theta_0 \subseteq \Theta$.
- Consider performing the following hypothesis test:

$$(H) : \begin{cases} H_0 : \theta \in \Theta_0. \\ H_1 : \theta \in \Theta_0^c, \end{cases}$$

- Let $(X_1, \ldots, X_n)$ denote the random sample and $\theta \mapsto L(\theta\|\mathbf{x})$ be the likelihood function.
- The **likelihood ratio test statistic** for testing (H) is

$$\lambda(\mathbf{x}) = \frac{\sup_{\theta \in \Theta_0} L(\theta\|\mathbf{x})}{\sup_{\theta \in \Theta} L(\theta\|\mathbf{x})}$$

Likelihood Ratio Tests

- In short **LRT**, a **likelihood ratio test** is any test ϕ that rejects $H_0 : \theta \in \Theta_0$ if $\lambda(\boldsymbol{x}) \leqslant k,\ \ k \in [0, 1]$.
- The constant k is determined from the size restriction:

$$\sup_{\theta \in \Theta_0} P_\theta \left(\lambda(\boldsymbol{X}) \leqslant k\right) = \alpha.$$

- If $\hat{\theta}$ is an MLE of θ obtained by doing an unrestricted maximization of $\theta \mapsto L(\theta \| \boldsymbol{x})$ and $\hat{\theta}_0$ is an MLE of θ obtained by doing an restricted maximization of $\theta \mapsto L(\theta \| \boldsymbol{x})$ over $\theta \in \Theta_0$, then

$$\lambda(\boldsymbol{x}) = \frac{L(\hat{\theta}_0 \| \boldsymbol{x})}{L(\hat{\theta} \| \boldsymbol{x})}$$

Methods of Finding Test Statistics
Likelihood Ratio Tests

Theorem

- *If T is a sufficient statistics for θ*
- *λ^* is an LRT statistic based on T*
- *and λ is an LRT statistic based on $\boldsymbol{X}$.*

Then, for any sample point $\boldsymbol{x}$, it holds that

$$\lambda^*[T(\boldsymbol{x})] = \lambda(\boldsymbol{x}).$$

Likelihood Ratio Tests

Example

A random sample $(X_1, \ldots, X_n)$ is drawn from an exponential population with pdf:

$$f_\theta(x) = \begin{cases} e^{-(x-\theta)} & x \geqslant \theta \\ 0 & x < \theta. \end{cases}$$

Consider performing the following hypothesis test.

$$(H) : \begin{cases} H_0 : \theta = \theta_0. \\ H_1 : \theta \neq \theta_0. \end{cases}$$

1. *Find the MLE of θ.*
2. *Find a size α the LRT for testing (H).*

Likelihood Ratio Tests

Example

A random sample $(X_1, \ldots, X_n)$ is drawn from $X \sim \mathcal{N}(\mu, 1)$. Consider performing the following hypothesis test:

$$(H): \begin{cases} H_0 : \mu = \mu_0. \\ H_1 : \mu \neq \mu_0. \end{cases}$$

1. *Find the MLE of μ.*
2. *Find a size α the LRT for testing (H).*

Likelihood Ratio Tests

Some Facts

- *LRTs are also useful in situations where there are <u>nuisance parameters</u>, that is parameters that are present in a model but that are not of direct inferential interest.*
- *The presence of nuisance parameters does not affect the LRT construction method but as might be expected, it leads to a different test.*

Likelihood Ratio Tests

Example

A random sample $(X_1, \ldots, X_n)$ is drawn from $X \sim \mathcal{N}(\mu, \sigma^2)$, with σ^2 unknown. Consider testing

$$(H) : \begin{cases} H_0 : \mu = \mu_0. \\ H_1 : \mu \neq \mu_0. \end{cases}$$

1. *Find the MLE of μ.*
2. *Find a size α the LRT for testing (H).*

Likelihood Ratio Tests

Exercise

Suppose that we have two independent random samples: $X_1, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim} \mathcal{E}(\theta)$ *and* $Y_1, \ldots, Y_m \overset{\perp\!\!\!\perp}{\sim} \mathcal{E}(\mu)$.

1. *Find the LRT of* $H_0 : \theta = \mu$ *versus* $H_1 : \theta \neq \mu$.
2. *Show that the test in part (1) can be based on*

$$T = \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i + \sum_{j=1}^{m} Y_j}.$$

3. *Find the distribution of* T *when* H_0 *is true.*

Most Powerful (MP) Tests

- In the level α class, the type-I error probabilities tests are controlled. They are at most α for all $\theta \in \Theta_0$.
- An optimal test in such a class would also have a small type-II error probability, that is, a large power function for $\theta \in \Theta_1$.
- If one test had a smaller type-II error probability than all other tests in the class, it would certainly be a strong contender for the best test in the class, a notion that is formalized in the next definition.

Most Powerful (MP) Tests

Definition

Let Φ_α be the class of all tests for the problem $(\alpha, \Theta_0, \Theta_1)$. A test $\phi_0 \in \Phi_\alpha$ is said to be a **most powerful (MP)** *against an alternative $\theta \in \Theta_1$ if*

$$\beta_{\phi_0}(\theta) \geqslant \beta_\phi(\theta), \quad \forall\, \phi \in \Phi_\alpha. \tag{1}$$

Remarks

- *Notice ϕ_0 may depend on the choice of $\theta \in \Theta_1$. In fact, if Θ_1 contains only one point, this definition suffices.*
- *But, if Θ_1 contains at least two points, as will usually be the case, we will have an MP test corresponding to each $\theta \in \Theta_1$.*

Uniformly Most Powerful (UMP) Tests

Definition

Let Φ_α be the class of all tests for the problem $(\alpha, \Theta_0, \Theta_1)$. A test $\phi_0 \in \Phi_\alpha$ is said to be a **uniformly most powerful** *if*

$$\beta_{\phi_0}(\theta) \geqslant \beta_\phi(\theta), \ \forall\, \theta \in \Theta_1 \text{ and } \phi \in \Phi_\alpha. \tag{2}$$

Remarks

- *A UMP test ϕ is a level-α test which maximizes the power among all tests of level α.*
- *If $H_1 : \theta \in \Theta_1$ is simple, then a UMP test ϕ is simply called a* **most powerful (MP)** *test.*

Uniformly Most Powerful (UMP) Tests

Reformulation (Hypothesis Testing Problem)

Let X be a random variable such that $X \sim f_\theta(x)$, $\theta \in \Theta$. is an unknown parameter. Consider testing:

$$H_0 : \theta \in \Theta_0 \text{ versus } H_1 : \theta \in \Theta_1$$

where Θ_0 is a subset of Θ and $\Theta_1 = \Theta_0^c$. Fix $\alpha \in [0,1]$. Given the sample point $\boldsymbol{x} = (x_1, x_2, ..., x_n)$, find a test $\phi : \mathbb{R}^n \to [0,1]$ such that:

$$\max_{\theta \in \Theta_1} \beta_\phi(\theta) \quad \text{subject to} \quad \begin{cases} \forall\ \theta \in \Theta_0, \\ \beta_\phi(\theta) \leqslant \alpha, \\ 0 \leqslant \alpha \leqslant 1. \end{cases}$$

Thus, if Θ_0 and Θ_1 are both composite, the problem is to find a UMP test ϕ for the problem $(\alpha, \Theta_0, \Theta_1)$.

Uniformly Most Powerful Tests

Theorem (Neyman-Pearson Lemma)

Let X be a random variable such that $X \sim f_\theta(x)$, $\theta \in \Theta$.

- f_{θ_0} *and* f_{θ_1} *be densities of* P_{θ_0} *and* P_{θ_1} *with respect to a* σ*-finite measure* μ.
- *Let* $0 \leqslant \alpha \leqslant 1$. *Consider testing the hypothesis problem*

$$(H):\begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta = \theta_1. \end{cases} \tag{3}$$

Uniformly Most Powerful Tests

Theorem (Neyman-Pearson Lemma)

i) *Any test ϕ of the following form (4):*

$$\phi(\mathbf{x}) = \begin{cases} 1, & f_{\theta_1}(\mathbf{x}) > kf_{\theta_0}(\mathbf{x}) \\ \gamma, & f_{\theta_1}(\mathbf{x}) = kf_{\theta_0}(\mathbf{x}) \\ 0, & f_{\theta_1}(\mathbf{x}) < kf_{\theta_0}(\mathbf{x}), \end{cases} \tag{4}$$

where $k \geqslant 0$, and $0 \leqslant \gamma \leqslant 1$ is MP of size $\beta_\phi(\theta_0) = \alpha$.

ii) *For any $0 < \alpha \leqslant 1$, there exists a test ϕ of the form (4) with $k \geqslant 0$, and $0 \leqslant \gamma \leqslant 1$.*

iii) *Furthermore, if ϕ^* is MP of size $0 < \alpha \leqslant 1$, then it has the form (4) a.e P_{θ_0} and P_{θ_1}. That is,* $P_{\theta_0}(\phi \neq \phi^*, f_{\theta_1} \neq f_{\theta_0}) = P_{\theta_1}(\phi \neq \phi^*, f_{\theta_1} \neq f_{\theta_0}) = 0$

Uniformly Most Powerful Tests

Remarks

- *Neyman-Pearson lemma applies to hypothesis tests:*

$$(H):\begin{cases} H_0 : X \sim f_0 \\ H_1 : X \sim f_1. \end{cases}$$

  where f_0 and f_1 are two probability density functions.

- *It follows from the proof of this fundamental result that the test ϕ in the Neyman-Pearson Lemma is MP even if f_1 and f_0 are not necessarily densities.*

Uniformly Most Powerful Tests

Corollary

Let X be a random variable such that $X \sim f_\theta(x)$, $\theta \in \Theta$. Consider performing the following hypothesis test:

$$(H): \begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta = \theta_1. \end{cases}$$

If $T(\boldsymbol{X})$ is a sufficient statistic for θ and $T \sim g(t\|\theta)$, then

$$\phi(t) = \begin{cases} 1, & g(t\|\theta_1) > kg(t\|\theta_0) \\ \gamma, & g(t\|\theta_1) = kg(t\|\theta_0) \\ 0, & g(t\|\theta_1) < kg(t\|\theta_0). \end{cases}$$

is a most powerful (MP) test with $E_{\theta_0}(\phi(T)) = \alpha$.

Uniformly Most Powerful Tests

Exercise

Consider testing the following statistical problem:

$$(H): \begin{cases} H_0 : X \sim \mathcal{N}(0,1) \\ H_1 : X \sim \mathcal{C}(0,1). \end{cases}$$

1. *Find a most powerful (MP) size α test of (H)*
2. *Find the power test against the alternative.*

Uniformly Most Powerful Tests

Exercise

Let $(X_1, \ldots, X_n)$ be a random sample from $X \sim \mathcal{B}(1, p)$.

1. *Find a UMP-α test for testing the following problem:*

$$(H) : \begin{cases} H_0 : p = p_0 \\ H_1 : p > p_0. \end{cases}$$

2. *Find a UMP-α test for testing the following problem:*

$$(H) : \begin{cases} H_0 : p = p_0 \\ H_1 : p < p_0. \end{cases}$$

Uniformly Most Powerful Tests

Exercise

Let $(X_1, \ldots, X_n)$ be a random sample from $X \sim \mathcal{N}(\mu, \sigma^2)$. Assume that μ_0 and σ_0^2 are known.

1. *Find a UMP-α test for testing the following problem:*

$$(H) : \begin{cases} H_0 : \mu = \mu_0,\ \sigma_0^2 > 0 \\ H_1 : \mu > \mu_0,\ \sigma_0^2 > 0. \end{cases}$$

2. *Find a UMP-α test for testing the following problem:*

$$(H) : \begin{cases} H_0 : \mu = \mu_0,\ \sigma_0^2 > 0 \\ H_1 : \mu < \mu_0,\ \sigma_0^2 > 0. \end{cases}$$

3. *Does a UMP-α test exist for testing these two problems if σ^2 is unknown?*

Uniformly Most Powerful Tests

Definition (MLR Property)

Let $\Theta \subseteq \mathbb{R}$. A univariate class of pdfs or pmfs

$$\{f_\theta : \theta \in \Theta\}$$

is said to have a **monotone likelihood ratio (MLR)** *in a statistic $T : \mathbb{R}^n \to \mathbb{R}$ if for every $\theta_1 < \theta_2$, whenever f_{θ_1} and f_{θ_2} are distinct, the likelihood ratio*

$$\boldsymbol{x} \mapsto \frac{f_{\theta_2}(\boldsymbol{x})}{f_{\theta_1}(\boldsymbol{x})} \text{ is monotone function on}$$

the set of values for which at least one of the $f_{\theta_1}, f_{\theta_2} > 0$, i.e

$$\{\boldsymbol{x} : f_{\theta_1}(\boldsymbol{x}) > 0 \text{ or } f_{\theta_2}(\boldsymbol{x}) > 0\}.$$

Remarks

- *The likelihood ratio is well defined if*

$$f_{\theta_1}(\boldsymbol{x}) > 0 \text{ or } f_{\theta_2}(\boldsymbol{x}) > 0.$$

  In other words, f_{θ_1} *and* f_{θ_2} *are not both* 0. *If* $f_{\theta_1}(\boldsymbol{x}) = 0$ *and* $f_{\theta_2}(\boldsymbol{x}) > 0$, *then the likelihood ratio* $\frac{f_{\theta_2}(\boldsymbol{x})}{f_{\theta_1}(\boldsymbol{x})}$ *is* ∞.

- *The likelihood ratio* $\frac{f_{\theta_2}(\boldsymbol{x})}{f_{\theta_1}(\boldsymbol{x})}$ *is nondecreasing in* $T(\boldsymbol{x})$ *if and only if it is nonincreasing in* $-T(\boldsymbol{x})$. *As such, showing that a family of distributions has a MLR property will reduce to checking its likelihood ratio is nondecreasing in* $T(\boldsymbol{x})$. *In other words, families of densities with nonincreasing MLR property in* $T(\boldsymbol{x})$ *can be treated by symmetry.*

Uniformly Most Powerful Tests

Exercise

Let X be one observation from a Cauchy$(0, \theta)$ with $\theta > 0$.

1. *Show that this family does not have an MLR property.*
2. *Show that $|X|$ is sufficient for θ and that the distribution of $|X|$ does have an MLR property.*

Lemma (Exponential class)

Let X be a random variable with a pdf/pmf such that

$$f_\theta(x) = h(x)c(\theta)e^{w(\theta)T(x)}, \ \theta \in \Theta, x \in \mathcal{X}.$$

Then, $\{f_\theta : \theta \in \Theta\}$ possesses the MLR property in $T(x)$ if and only if $\theta \mapsto w(\theta)$ is nondecreasing.

Uniformly Most Powerful Tests

Theorem (Karlin Rubin)

Let $X \sim f_\theta, \theta \in \Theta$, where $\{f_\theta\}$ has an MLR in $T : \mathbb{R}^n \to \mathbb{R}$. For testing $H_0 : \theta \leqslant \theta_0$ against $H_1 : \theta > \theta_0$, $\theta_0 \in \Theta$, any test

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & T(\boldsymbol{x}) > t_0, \\ \gamma, & T(\boldsymbol{x}) = t_0, \\ 0, & T(\boldsymbol{x}) < t_0, \end{cases} \tag{5}$$

i) *has a nondecreasing power function β_ϕ;*

ii) *is UMP of its size $E_{\theta_0}\phi(\boldsymbol{X}) = \alpha$.*

iii) *Moreover, for every $0 \leqslant \alpha \leqslant 1$ and every $\theta_0 \in \Theta$, there exist a $t_0 \in \mathbb{R}$ and $0 \leqslant \gamma \leqslant 1$ such that ϕ in (6) is the UMP size α test of H_0 against H_1.*

Uniformly Most Powerful Tests

Exercise

Let X_i be the number of arrivals at service counter in the i-th day of a study. Assume that X_i are distributed according to a Poisson distribution with the arrival rate of θ persons per day. Derive the UMP-α test for this testing problem.

$$(H): \begin{cases} H_0 : \theta \leqslant \theta_0. \\ H_1 : \theta > \theta_0. \end{cases}$$

Remark

By interchanging inequalities throughout in the preceding Theorem , we see that this theorem also provides a solution of the dual statistical testing problem:

$$(H): \begin{cases} H_0 : \theta \geqslant \theta_0 \\ H_1 : \theta < \theta_0. \end{cases}$$

Uniformly Most Powerful Tests

Theorem (Karlin-Rubin)

Let X be a random variable such that $X \sim f_\theta(x)$, $\theta \in \Theta$. Consider performing one of the following hypothesis tests:

$$H_0 : \theta \leqslant \theta_0 \text{ versus } H_1 : \theta > \theta_0.$$

$$H_0 : \theta \geqslant \theta \text{ versus } H_1 : \theta < \theta_0.$$

Let $T : \mathbb{R}^n \to \mathbb{R}$ be a sufficient statistic for θ such that $T \sim g_\theta(t)$. Further, the class of univariate pmfs or pfds

$$\{g_\theta : \theta \in \Theta\}$$

have monotone likelihood ratio (MLR) property.

Uniformly Most Powerful Tests

Theorem (Karlin-Rubin (Cont'd))

Then for any t_0,

- *if $H_0 : \theta \leqslant \theta_0$ versus $H_1 : \theta > \theta_0$, then*

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & T(\boldsymbol{x}) > t_0, \\ \gamma, & T(\boldsymbol{x}) = t_0, \\ 0, & T(\boldsymbol{x}) < t_0, \end{cases} \tag{6}$$

  the test that rejects H_0 if and only if $T > t_0$ is a UMP level α test, where $\alpha = P_{\theta_0}(T > t_0)$;

- *if $H_0 : \theta \geqslant \theta$ versus $H_1 : \theta < \theta_0$, then the test that rejects H_0 if and only if $T < t_0$ is a UMP level α test, where $\alpha = P_{\theta_0}(T > t_0)$.*

Uniformly Most Powerful Tests

Exercise

Let $X_1, X_2, \ldots, X_n \sim \mathcal{U}[0, \theta]$, $\theta > 0$.

1. *Does the family have an MLR property?*
2. *Find a UMP-α test for testing*

$$(H) : \begin{cases} H_0 : \theta \leqslant \theta_0. \\ H_1 : \theta > \theta_0. \end{cases}$$

3. *Find the power of the test ϕ in (1) against H_1.*

Uniformly Most Powerful Tests

Exercise

Let F and G be two known cdfs on $\mathbb{R}$ and X be a single observation from the cdf H such that $\forall\ x \in \mathbb{R}$,

$$H(x) = \theta F(x) + (1-\theta)G(x),$$

where $\theta \in [0,1]$ is unknown. Find a UMP-α test for testing

$$(H) : \begin{cases} H_0 : \theta \leqslant \theta_0. \\ H_1 : \theta > \theta_0. \end{cases}$$

Uniformly Most Powerful Tests

Exercise

Let X be random variable with hypergeometric pmf

$$P_m(X = x) = \frac{\binom{m}{x}\binom{M-m}{n-x}}{\binom{M}{n}}, \quad x = 0, 1, 2, \ldots, m.$$

Find a UMP-α test for testing the statistical problem

$$(H) : \begin{cases} H_0 : m \leqslant m_0. \\ H_1 : m > m_0. \end{cases}$$

Uniformly Most Powerful Tests

Exercise

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim}$ Pareto$(1, \theta)$. Let μ denote its mean.

1. *Express μ in terms of θ.*
2. *Find a UMP test for testing*

$$(H) : \begin{cases} H_0 : \mu = \mu_0. \\ H_1 : \mu < \mu_0. \end{cases}$$

3. *Find the distribution of the test statistic found in (2) under $H_0 : \mu = \mu_0$.*

Uniformly Most Powerful Tests

Theorem

Suppose that X is a rv with pmf/pdf such that

$$f_\theta(x) = h(x)c(\theta)e^{w(\theta)T(x)}, \ \theta \in \Theta.$$

Let $\theta_1 < \theta_2$. There exists a UMP test of the hypothesis

$$(H) : \begin{cases} H_0 : \theta \leqslant \theta_1 \text{ or } \theta \geqslant \theta_2 \\ H_1 : \theta_1 < \theta < \theta_2. \end{cases}$$

$$\phi(\mathbf{x}) = \begin{cases} 1, & k_1 < T(\mathbf{x}) < k_2, \\ \gamma_i & T(\mathbf{x}) = k_i, \quad i = 1, 2 \\ 0, & T(\mathbf{x}) < k_1 \ or \ > k_2, \end{cases}$$

where the $k_1 < k_2$'s and the γ's are given by

$$E_{\theta_1}\phi(\mathbf{X}) = E_{\theta_2}\phi(\mathbf{X}) = \alpha.$$

Uniformly Most Powerful Tests

Exercise

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim} \mathcal{N}(\mu, 1)$. *Let* $\mu_1 > \mu_0$.
Find a UMP-α *test for testing*

$$(H) : \begin{cases} H_0 : \mu \leqslant \mu_0 \text{ or } \mu \geqslant \mu_1. \\ H_1 : \mu_0 < \mu < \mu_1. \end{cases}$$

Uniformly Most Powerful Tests

Exercise

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim} \mathcal{B}(1, \theta)$. *Let* $\theta_1 > \theta_0$.

1. *Find a UMP-*α *test for testing*

$$(H) : \begin{cases} H_0 : \theta \leqslant \theta_0 \text{ or } \theta \geqslant \theta_1. \\ H_1 : \theta_0 < \theta < \theta_1. \end{cases}$$

   if $\theta_0 = 1/4,\ \theta_1 = 3/4,\ \alpha = 0.05,\ n = 25,$
   $k_1 = 10,\ k_2 = 15.$

2. *Find the power against the alternative* $\theta = 1/2$.

Uniformly Most Powerful Tests

Remarks

- *Note that UMP tests for testing one of the following:*
  i) $H_0 : \theta_1 \leqslant \theta \leqslant \theta_2$ *vs* $H_1 : \theta \leqslant \theta_1$ *or* $\theta \geqslant \theta_2$.
  ii) $H_0 : \theta = \theta_0$ *versus* $H_1 : \theta \neq \theta_0$

  for the one-parameter exponential family do not exist.
- *When a UMP test does not exist, we may use the same approach used in estimation problems, that is imposing a reasonable restriction on the tests to be considered and finding optimal tests within the class of tests under restriction. Two such types of restrictions in estimation problems are* **unbiasedness** *and* **invariance**.

Uniformly Most Powerful Tests

Exercise

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim} \mathcal{N}(0, \sigma^2)$.

1. *Find a UMP test for testing*

$$(H) : \begin{cases} H_0 : \sigma \leqslant \sigma_0. \\ H_1 : \sigma > \sigma_0. \end{cases}$$

2. *Find a UMP test for testing*

$$(H) : \begin{cases} H_0 : \sigma \geqslant \sigma_0. \\ H_1 : \sigma < \sigma_0. \end{cases}$$

3. *Consider testing* $H_0 : \sigma = \sigma_0$ *against* $H_1 : \sigma \neq \sigma_0$. *Show that a UMP test of* H_0 *does not exist.*

Uniformly Most Powerful Unbiased Tests

Definition

Let Φ_α be the class of all tests for the problem $(\alpha, \Theta_0, \Theta_1)$. A test ϕ is said to be a **unbiased** *if*

$$\beta_\phi(\theta_0) \leqslant \beta_\phi(\theta_1), \ \forall\, \theta_0 \in \Theta_0, \theta_1 \in \Theta_1.$$

Definition

Let Φ_α be the class of all tests for the problem $(\alpha, \Theta_0, \Theta_1)$. A test ϕ is said to be a **unbiased** *if $\beta_\phi(\theta) \geqslant \alpha, \ \forall\, \theta \in \Theta_1$.*

Uniformly Most Powerful Unbiased Tests

Remarks

It follows that an unbiased test ϕ is such that:

$$\begin{cases} \beta_\phi(\theta) \leqslant \alpha, & \forall\ \theta \in \Theta_0 \\ \beta_\phi(\theta) \geqslant \alpha & \forall\ \theta \in \Theta_1. \end{cases}$$

The defining property of an **unbiased test** *is that the type-I error probability is at most α and the power of the test is at least α. This means that ϕ is at least as good as the trivial test $\phi \equiv \alpha$. An unbiased test rejects a false H_0 more often than a true H_0.*

Uniformly Most Powerful Unbiased Tests

Exercise

Let $X_1, X_2, \ldots, X_n$ be i.i.d rvs with pdf:

$$f(x\|\lambda) = 2\lambda x e^{-\lambda x^2}, \; x \geq 0, \; \lambda > 0.$$

Show that the statistical test function

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & 2\lambda_0 \sum_{i=1}^{n} X_i^2 > \mathcal{X}^2_{2n,\alpha}, \\ 0 & \text{otherwise} \end{cases}$$

is unbiased for testing the statistical problem

$$(H) : \begin{cases} H_0 : \frac{1}{\lambda} \leq \frac{1}{\lambda_0} \\ H_1 : \frac{1}{\lambda} > \frac{1}{\lambda_0}. \end{cases}$$

Uniformly Most Powerful Unbiased Tests

Exercise

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim} \mathcal{N}(\theta, 1)$. *Consider testing*

$$(H) : \begin{cases} H_0 : \theta \leqslant \theta_0 \\ H_1 : \theta > \theta_0. \end{cases}$$

Let $\alpha \in [0, 1]$. *The test function* ϕ *for testing* (H) *is*

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \bar{x} > \theta_0 + \frac{z_\alpha}{\sqrt{n}} \\ 0 & \textit{otherwise.} \end{cases}$$

Show that the statistical test function ϕ *is unbiased.*

Uniformly Most Powerful Unbiased Tests

Definition

Let U_α be the class of all unbiased size α tests of H_0. If there exists a test $\phi \in U_\alpha$ that has maximum power at each $\theta \in \Theta_1$, we call ϕ a **UMP unbiased** *size α test.*

Remark

Clearly $U_\alpha \subset \Phi_\alpha$. If a UMP test exists in Φ_α, it is UMP in U_α. This follows by comparing the power of the UMP test with that of the trivial test $\phi(x) = \alpha$. It is convenient to introduce another class of tests.

Uniformly Most Powerful Unbiased Tests

Lemma

If ϕ is a unbiased test function for the hypothesis testing problem $(\alpha, \Theta_0, \Theta_1)$ and T is a sufficient statistic for θ, then

- $E_\theta(\phi \| T)$ *is unbiased test function for* $(\alpha, \Theta_0, \Theta_1)$.
- ϕ *and* $E_\theta(\phi \| T)$ *have the same power.*

Remark

As a result, attention can be restricted to

- *either* **tests based on the sufficient statistic**
- *or* **tests that are functions of the sufficient statistic**.

in the process of finding UMPU tests.

Uniformly Most Powerful Unbiased Tests

Definition

*Let X be a random variable such that $X \sim f_\theta(x)$, $\theta \in \Theta$. Consider the hypothesis testing problem $(\alpha, \Theta_0, \Theta_1)$. A test ϕ is said to be α-**similar** on a subset Θ^* of Θ if*

$$\beta_\phi(\theta) = E_\theta \phi(\mathbf{X}) = \alpha, \quad \forall\ \theta \in \Theta^*$$

Remark

It is clear that there exists at least one α-similar test on every Θ^*, namely, $\phi(x) \equiv \alpha$, $0 \leqslant \alpha \leqslant 1$.

Uniformly Most Powerful Unbiased Tests

Theorem

Let X be a random variable such that $X \sim f_\theta(x)$, $\theta \in \Theta$. Consider testing the following statistical problem:

$$(H): \begin{cases} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_1. \end{cases}$$

Let $\theta \mapsto \beta_\phi(\theta)$ be continuous for any ϕ. If ϕ is an unbiased size α test of (H), then, it is α-similar on the boundary

$$\Lambda = \overline{\Theta}_0 \cap \overline{\Theta}_1,$$

that is, the set of points θ that are limit points of both Θ_0 and Θ_1. Recall that $\overline{A}$ is the closure of set A.

Uniformly Most Powerful Unbiased Tests

Lemma

If $\theta \mapsto \beta_\phi(\theta)$ is continuous, then for any unbiased size α test ϕ for testing the statistical problem $(\alpha, \Theta_0, \Theta_1)$ is also α-similar for the pdfs (pmfs) of Λ, that is, for

$$\{f_\theta : \theta \in \Lambda\}.$$

Remark

If we can find an MP similar test of $H_0 : \theta \in \Lambda$ against H_1, and if this test is unbiased size α, then necessarily it is MP in the smaller class.

Uniformly Most Powerful Unbiased Tests

Definition

A test ϕ that is UMP among all α-similar tests on the boundary $\Lambda = \overline{\Theta}_0 \cap \overline{\Theta}_1$ is said to be a **UMP α-similar test**.

Theorem

Let X be a random variable such that $X \sim f_\theta(x)$, $\theta \in \Theta$. Consider testing the following statistical problem:

$$(H) : \begin{cases} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_1. \end{cases}$$

Let $\theta \mapsto \beta_\phi(\theta)$ be continuous for any ϕ. Then a UMP α-similar test is UMP unbiased, provided that its size is α for testing H_0 against H_1.

Uniformly Most Powerful Unbiased Tests

Remarks

- *The rationale behind this theorem is that tests that are UMP similar on the boundary are often UMP unbiased.*
- *Note that if the family of pdf/pmf f_θ is in exponential family then the power function $\theta \mapsto \beta_\phi(\theta)$ is continuous.*

Exercise

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim} \mathcal{N}(\theta, 1)$.
Find a UMPU test for the hypothesis testing:

$$(H) : \begin{cases} H_0 : \theta \leqslant 0. \\ H_1 : \theta > 0. \end{cases}$$

Uniformly Most Powerful Invariant Tests

Let $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ be the class of pdfs or pmfs of X. Let $(\mathcal{G}, \circ)$ be a group of transformations $g : \mathcal{X} \to \mathcal{X}$ on the space of values $\mathcal{X}$ of X, that is:

(i) $\forall\ g_1, g_2 \in \mathcal{G},\ g_1 \circ g_2 \in \mathcal{G}$.

(ii) $\forall\ g_1, g_2, g_3 \in \mathcal{G},\ (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

(iii) $\exists\ e \in \mathcal{G}$ such that $g \circ e = e \circ g = g,\ \forall\ g \in \mathcal{G}$.

(iv) $\forall\ g \in \mathcal{G},\ \exists\ f \in \mathcal{G}$ such that $g \circ f = f \circ g = e$.

Consider testing the statistical hypothesis problem:

$$(H) : \begin{cases} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_1. \end{cases}$$

Uniformly Most Powerful Invariant Tests

As such, we need to reformulate the principle of invariance.

- First, we need to ensure that under transformations $\mathcal{G}$
  - i) not only does $\mathcal{F}$ remain invariant
  - ii) but also the problem of testing (H) remain invariant.
- Second, since the problem has not changed by application of $\mathcal{G}$, the decision rule ϕ also must not change.

In what follows, we introduce invariance under $\mathcal{G}$ of a family of pdfs/pmfs or a family of distributions,

Uniformly Most Powerful Invariant Tests

Definition

Let $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ be the class of pdfs or pmfs of X.
Let $(\mathcal{G}, \circ)$, with $\mathcal{G} = \{g : \mathcal{X} \to \mathcal{X}\}$ be a group of transformations on the space $\mathcal{X}$ of values of X.
A **class $\mathcal{F}$ is invariant under $\mathcal{G}$** *if*

$$\forall\ g \in \mathcal{G},\ \ \theta \in \Theta;\ \exists!\ \theta' = \overline{g}\theta \in \Theta \text{ such that}$$
$$X \sim f_\theta(x) \Rightarrow g(X) \sim f_{\theta'}(x).$$

Uniformly Most Powerful Invariant Tests

Definition

Let $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ *be the class of pdfs or pmfs of* X*.*
Let $(\mathcal{G}, \circ)$*, with* $\mathcal{G} = \{g : \mathcal{X} \to \mathcal{X}\}$ *be a group of transformations on the space* $\mathcal{X}$ *of values of* X*.*
We say that $\mathcal{G}$ *leaves a* **hypothesis testing problem**

$$(H) : \begin{cases} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_1. \end{cases}$$

invariant *if* $\mathcal{G}$ *leaves invariant the following two classes:*

i) $\mathcal{F}_0 = \{f_\theta : \theta \in \Theta_0\}$

ii) $\mathcal{F}_1 = \{f_\theta : \theta \in \Theta_1\}$.

Uniformly Most Powerful Invariant Tests

Exercise

Let $X \sim \mathcal{B}(n, \theta)$, *where* $0 \leqslant \theta \leqslant 1$. *Let* $\mathcal{G} = \{g_1, g_2\}$ *be a set of transformations on the space of values of* X, *where* $\forall\ x,\ g_1(x) = x$ *and* $g_2(x) = n - x$. *Consider testing*

$$(H) : \begin{cases} H_0 : \theta = \frac{1}{2} \\ H_1 : \theta \neq \frac{1}{2}. \end{cases}$$

1. *Show that* $(\mathcal{G}, \circ)$ *is a group.*
2. *Show that* (H) *is invariant under* $\mathcal{G}$.

Uniformly Most Powerful Invariant Tests

Definition

Let $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ be the class of pdfs or pmfs of X. Let I_α denote the class of all invariant size α tests of

$$(H) : \begin{cases} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_1. \end{cases}$$

If there exists a UMP member in I_α, we call the test a **UMP invariant test** *of H_0 against H_1.*

Remark

The search for UMP invariant tests is greatly facilitated by the use of the following result.

Uniformly Most Powerful Invariant Tests

Definition

Let $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ be the class of pdfs or pmfs of X. Let $(\mathcal{G}, \circ)$, with $\mathcal{G} = \{g : \mathcal{X} \to \mathcal{X}\}$ be a group of transformations on the space $\mathcal{X}$ of values of X. Consider testing:

$$(H) : \begin{cases} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_1. \end{cases}$$

A **statistical test ϕ is invariant under $\mathcal{G}$** *if*

$$\forall\, \mathbf{x} \in \mathcal{X}, g \in \mathcal{G},\ \ \phi[g(\mathbf{x})] = \phi(\mathbf{x}).$$

Remark

An invariant ϕ is a decision rule that does not depend on the measurement scale $g(\mathbf{x})$ that is used.

Uniformly Most Powerful Invariant Tests

Definition

Let $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ be the class of pdfs or pmfs of X. Let $(\mathcal{G}, \circ)$, with $\mathcal{G} = \{g : \mathcal{X} \to \mathcal{X}\}$ be a group of transformations on the space $\mathcal{X}$ of values of X. Consider testing:

$$(H): \begin{cases} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_1. \end{cases}$$

A statistic $T : \mathbb{R}^n \to \mathbb{R}$ is **maximal invariant** *under $\mathcal{G}$ if*

i) *T is invariant; i.e $\forall\, \mathbf{x} \in \mathcal{X}, g \in \mathcal{G},\ T[g(\mathbf{x})] = T(\mathbf{x})$.*

ii) *T is maximal, i.e $T(\mathbf{x}) = T(\mathbf{y}) \Rightarrow\ \exists\, g \in \mathcal{G} : \mathbf{x} = g(\mathbf{y})$.*

Uniformly Most Powerful Invariant Tests

Exercise

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$. *Define*

$$\mathcal{G} = \left\{ g_\alpha \in \mathbb{R}^{\mathbb{R}^n} : \forall\, \boldsymbol{x} \in \mathbb{R}^n,\ g_\alpha(\boldsymbol{x}) = (x_1 + \alpha, \ldots, x_n + \alpha) \right\},$$

the group of translations and consider the statistic

$$\forall\, \boldsymbol{x} \in \mathbb{R}^n,\ T(\boldsymbol{x}) = (x_n - x_1, \ldots, x_n - x_{n-1})$$

Show that $T : \mathbb{R}^n \to \mathbb{R}$ *such that is a maximal invariant.*

Uniformly Most Powerful Invariant Tests

Exercise

Let $a > 0$, $b \in \mathbb{R}$ and the group of translation and scale:

$$\mathcal{G} = \{g_{a,b} | \ \forall \, \boldsymbol{x} \in \mathbb{R}^n, \ g_{a,b}(x) = (ax_1 + b, \ldots, ax_n + b)\},$$

$$\bar{x} = \frac{1}{n}\sum_1^n x_i \text{ and } \beta = \frac{1}{n}\sum_1^n (x_i - \bar{x})^2.$$

$$\forall \, \boldsymbol{x} \in \mathbb{R}^n, \ T_\beta(\boldsymbol{x}) = \begin{cases} 0 & \beta = 0, \\ \left(\frac{x_1 - \bar{x}}{\beta}, \frac{x_2 - \bar{x}}{\beta}, \ldots, \frac{x_n - \bar{x}}{\beta}\right) & \beta \neq 0 \end{cases}.$$

Show that $T_\beta : \mathbb{R}^n \to \mathbb{R}$ is a maximal invariant statistic.

Uniformly Most Powerful Invariant Tests

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}$ be maximal invariant with respect to a group $\mathcal{G}$ of transformations on the space of values of X, then,

$$\phi \text{ is invariant under } \mathcal{G} \Leftrightarrow \phi \; \exists \; h : \phi = h(T).$$

Remark

The take away from this result is that if a hypothesis testing problem is invariant under a group $\mathcal{G}$, then the principle of invariance restricts attention to invariant tests.
Therefore, it suffices to restrict attention to test functions that are functions of maximal invariant T.

Uniformly Most Powerful Invariant Tests

Exercise

Consider testing the following statistical problem:

$$(H):\begin{cases} H_0: X \sim \mathcal{N}(\theta, 1) \\ H_1: X \sim \mathcal{C}(\theta, 1). \end{cases}$$

Find a UMP invariant size α test of (H).

Uniformly Most Powerful Invariant Tests

Exercise

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim} \mathcal{N}(\mu, \sigma^2)$*, with* μ *and* σ^2*-unknown. Consider testing the following statistical problem*

$$(H) : \begin{cases} H_0 : \sigma^2 \leqslant \sigma_0^2. \\ H_1 : \sigma^2 > \sigma_0^2. \end{cases}$$

Find a UMPI-α *test of* (H)*.*

Locally Most Powerful Tests

- In the previous sections, we argued that whenever a UMP test does not exist, we restrict the class of tests under consideration and then find a UMP test in the subclass.
- Yet another approach when no UMP test exists is to restrict the parameter set to a subset of Θ_1. Tests that have good power properties for "local alternatives" may also retain good power properties for "nonlocal" alternatives.

Locally Most Powerful Tests

Definition

Let $\Theta \subseteq \mathbb{R}$. *Then a test* ϕ_0 *with power function*

$$\beta_{\phi_0}(\theta) = E_\theta \phi_0(\boldsymbol{X})$$

is said to be a **locally most powerful (LMP)** *test of*

$$(H): \begin{cases} H_0 : \theta \leqslant \theta_0 \\ H_1 : \theta > \theta_0. \end{cases}$$

if there exists $\epsilon > 0$ *such that for any other test* ϕ *with*

$$\beta_\phi(\theta_0) = \beta_{\phi_0}(\theta_0) = \int \phi(\boldsymbol{x}) f_{\theta_0}(\boldsymbol{x}) d\boldsymbol{x} \qquad (7)$$

$$\beta_{\phi_0}(\theta) \geqslant \beta_\phi(\theta), \ \forall\, \theta \in (\theta_0, \theta_0 + \epsilon].$$

Locally Most Powerful Tests

Lemma

We assume that the tests under consideration have continuously differentiable power function at $\theta = \theta_0$ and the derivative may be taken under the integral sign. In that case, an LMP test maximizes

$$\frac{\partial}{\partial\theta}\beta_\phi(\theta)\bigg|_{\theta=\theta_0} = \beta'_\phi(\theta)\big|_{\theta=\theta_0} = \int \phi(\boldsymbol{X})\frac{\partial}{\partial\theta}f_\theta(\boldsymbol{x})\bigg|_{\theta=\theta_0} d(\boldsymbol{x})$$

subject to the size constraint (7). A slight extension of the Neyman–Pearson lemma implies that a test satisfying (7) and given by

Locally Most Powerful Tests

Lemma (Cont'd)

$$\phi_0(\boldsymbol{x}) = \begin{cases} 1 & \textit{if } \left.\frac{\partial}{\partial\theta} f_\theta(\boldsymbol{x})\right|_{\theta=\theta_0} > k f_\theta(\boldsymbol{x}), \\ \gamma & \textit{if } \left.\frac{\partial}{\partial\theta} f_\theta(\boldsymbol{x})\right|_{\theta=\theta_0} = k f_\theta(\boldsymbol{x}), \\ 0, & \textit{if } \left.\frac{\partial}{\partial\theta} f_\theta(\boldsymbol{x})\right|_{\theta=\theta_0} > k f_\theta(\boldsymbol{x}), \end{cases}$$

will maximize $\beta'_\phi(\theta_0)$.

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Locally Most Powerful Tests

Remark

Note that for $\mathbf{x}$ *for which* $f_{\theta_0}(\mathbf{x}) \neq 0$ *we can write*

$$\phi_0(\mathbf{x}) = \begin{cases} 1 & \text{if } \left.\frac{\partial}{\partial\theta} \log f_\theta(\mathbf{x})\right|_{\theta=\theta_0} > k, \\ \gamma & \text{if } \left.\frac{\partial}{\partial\theta} \log f_\theta(\mathbf{x})\right|_{\theta=\theta_0} = k, \\ 0, & \text{if } \left.\frac{\partial}{\partial\theta} \log f_\theta(\mathbf{x})\right|_{\theta=\theta_0} > k. \end{cases}$$

Locally Most Powerful Tests

Exercise

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim} \mathcal{N}(\theta, \sigma^2)$, *with* σ^2*- known. Consider the problem of testing hypothesis test:*

$$(H): \begin{cases} H_0 : \theta \leqslant 0. \\ H_1 : \theta > 0. \end{cases}$$

1. *Study the existence of UMP test for testing (H).*
2. *Find a LMP test for testing (H).*

Locally Most Powerful Tests

Exercise

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim} \mathcal{N}(\mu, \sigma^2)$, *with* μ*- known. Consider the problem of testing hypothesis test:*

$$(H) : \begin{cases} H_0 : \sigma \leqslant \sigma_0. \\ H_1 : \sigma > \sigma_0. \end{cases}$$

1. *Study the existence of UMP test for testing (H).*
2. *Find a LMP test for testing (H).*

Locally Most Powerful Tests

Exercise

Let $X_1, X_2, \ldots, X_n$ be iid RVs with common PDF

$$f_\theta(x) = \frac{1}{\pi[1 + (x - \theta)^2]}, \quad x \in \mathbb{R},$$

and consider the problem of testing hypothesis test:

$$(H) : \begin{cases} H_0 : \theta \leqslant 0. \\ H_1 : \theta > 0. \end{cases}$$

1. *Study the existence of UMP test for testing (H).*
2. *Find a LMP test for testing (H).*

Union-Intersection and Intersection-Union Tests

In some situations, tests for complicated null hypotheses can be developed from tests for simpler null hypotheses. In what follows, we present two related methods:

i) union-intersection method

ii) intersection-union method

Union-Intersection Tests

The **union-intersection method of test construction** might be useful when testing the following statistical problem:

$$(H) : \begin{cases} H_0 : \theta \in \bigcap_{\gamma \in \Gamma} \Theta_\gamma \\ H_1 : \theta \in \bigcup_{\gamma \in \Gamma} \Theta_\gamma^c \end{cases}$$

where Γ is an arbitrary index set that may be finite or infinite, depending on the problem.

Union-Intersection Tests

First, consider the hypothesis testing problems:

$$(H_\gamma) : \begin{cases} H_{0\gamma} : \theta \in \Theta_\gamma \\ H_{1\gamma} : \theta \in \Theta_\gamma^c \end{cases}$$

Suppose that the test functions of (H_γ) are available:

$$\phi_\gamma(\boldsymbol{x}) = \begin{cases} 1, & T_\gamma(\boldsymbol{x}) \in C_\gamma \\ 0, & \text{otherwise.} \end{cases}$$

Then, the union-intersection test ϕ of (H) is:

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \bigcup_{\gamma \in \Gamma} \{\boldsymbol{x} : T_\gamma(\boldsymbol{x}) \in C_\gamma\}. \\ 0, & \text{otherwise.} \end{cases}$$

Union-Intersection Tests

Remarks

- *The rationale is simple. If any one of the hypotheses $H_{0\gamma}$ is rejected, then H_0, which, by* (**??**)*, is true only if $H_{0\gamma}$ is true for every γ, must also be rejected.*
- *Only if each of the hypotheses $H_{0\gamma}$ is accepted as true will the intersection H_0 be accepted as true.*

Union-Intersection Tests

Remarks

In some situations a simple expression for the rejection region of a union-intersection test can be found. In particular, suppose that each of the individual tests has a rejection region of the form $\{\mathbf{x} : T_\gamma(\mathbf{x}) > c\}$, *where c does not depend on* γ. *The rejection region for the union-intersection test can be expressed as:*

$$\bigcup_{\gamma \in \Gamma} \{\mathbf{x} : T_\gamma(\mathbf{x}) > c\} = \{\mathbf{x} : sup_{\gamma \in \Gamma} T_\gamma(\mathbf{x}) > c\}.$$

Thus, the test statistic for testing H_0 *is*

$$T(\mathbf{x}) = sup_{\gamma \in \Gamma} T_\gamma(\mathbf{x}).$$

Union-Intersection Tests

Exercise

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim} \mathcal{N}(\mu, \sigma^2)$*, with* μ*- known.*
Consider the hypothesis testing problem:

$$(H) : \begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu \neq \mu_0, \end{cases}$$

where μ_0 *is a specified number.*
Find the union-intersection test of (H)*.*

Intersection-Union Test

Another method, the **intersection-union method**, may be useful when testing the following statistical problem:

$$(H):\begin{cases} H_0 : \theta \in \bigcup_{\gamma\in\Gamma} \Theta_\gamma \\ H_1 : \theta \in \bigcap_{\gamma\in\Gamma} \Theta_\gamma^c \end{cases}$$

First, consider the hypothesis testing problems:

$$(H_\gamma):\begin{cases} H_{0\gamma} : \theta \in \Theta_\gamma \\ H_{1\gamma} : \theta \in \Theta_\gamma^c \end{cases}$$

Intersection-Union Test

Suppose that the test functions for each $\gamma \in \Gamma$ is:

$$\phi_\gamma(\boldsymbol{x}) = \begin{cases} 1, & T_\gamma(\boldsymbol{x}) \in C_\gamma \\ 0, & \text{otherwise.} \end{cases}$$

Then, the test function ϕ for union-intersection of (H) is:

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \bigcap_{\gamma \in \Gamma} \{\boldsymbol{x} : T_\gamma(\boldsymbol{x}) \in C_\gamma\}. \\ 0, & \text{otherwise.} \end{cases}$$

Intersection-Union Test

Remarks

- *H_0 is false if and only if all of the $H_{0\gamma}$ are false, so H_0 can be rejected if and only if each of the individual hypotheses $H_{0\gamma}$ can be rejected.*
- *Again, the test can be greatly simplified if the rejection regions for the individual hypotheses are all of the form $\{\mathbf{x} : T_\gamma(\mathbf{x}) \geq c\}$ (c independent of γ). In such cases, the rejection region for H_0 is*

$$\bigcap_{\gamma \in \Gamma} \{\mathbf{x} : T_\gamma(\mathbf{x}) \geq c\} = \{\mathbf{x} : inf_{\gamma \in \Gamma} T_\gamma(\mathbf{x}) \geq c\}.$$

Here, the intersection-union test statistic is $inf_{\gamma \in \Gamma} T_\gamma(\mathbf{X})$, and the test rejects H_0 for large values of this statistic.

Bayesian Tests

- Let X be a random variable with the following **sampling distribution**:

$$X \sim f(x\|\theta), \quad \theta \in \Theta.$$

Consider performing the following hypothesis test:

$$(H) : \begin{cases} H_0 : \theta \in \Theta_0, \\ H_1 : \theta \in \Theta_0^c. \end{cases} \qquad \Theta_0 \subseteq \Theta.$$

- Unlike in the classical approach, in the Bayesian setting θ is considered as random variable whose subjective probability distribution is called **prior distribution**, i.e

$$\theta \sim \pi(\theta).$$

Bayesian Tests

- A sample $(X_1, \ldots, X_n)$ is drawn from the population indexed by θ, and the prior distribution is updated with this sample information. The updated prior is called the **posterior distribution**, that is

$$\theta \| (x_1, \ldots, x_n) \sim \pi(\theta \| x_1, \ldots, x_n)$$

- This can be used to calculate the probabilities that H_0 and H_1 are true, also known as **posterior probabilities**:

$$\begin{cases} P\left(\theta \in \Theta_0 \| x_1, \ldots, x_n\right) = P(H_0 \text{ true} \| x_1, \ldots, x_n) \\ P\left(\theta \in \Theta_0^c \| x_1, \ldots, x_n\right) = P(H_1 \text{ true} \| x_1, \ldots, x_n) \end{cases}$$

Bayesian Tests

- One way a Bayesian tester may choose to use the posterior distribution is to decide to reject $H_0 : \theta \in \Theta_0$ if

$$P\left(\theta \in \Theta_0 \| x_1, \ldots, x_n\right) < P\left(\theta \in \Theta_0^c \| x_1, \ldots, x_n\right)$$

- The **Bayesian test statistic** is

$$P\left(\theta \in \Theta_0^c \| X_1, \ldots, X_n\right)$$

- The **rejection region** of a bayesian test is:

$$\mathcal{R} = \left\{(x_1, \ldots, x_n) : P\left(\theta \in \Theta_0^c \| x_1, \ldots, x_n\right) > \frac{1}{2}\right\}$$

t-Tests

Let $X_1, X_2, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$, where σ^2 is known.

1. If $H_0 : \mu \leqslant \mu_0$ versus $H_1 : \mu > \mu_0$, then

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \bar{x} \geqslant \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$

2. If $H_0 : \mu \geqslant \mu_0$ versus $H_1 : \mu < \mu_0$ then

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \bar{x} \leqslant \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$

3. If $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$, then

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & |\bar{x} - \mu_0| \geqslant z_\alpha \frac{\sigma}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$

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t-Tests

Let $X_1, X_2, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$, where σ^2 is unknown.

1. If $H_0 : \mu \leqslant \mu_0$ versus $H_1 : \mu > \mu_0$, then

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \bar{x} \geqslant \mu_0 + t_{n-1,\alpha} \frac{s}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$

2. If $H_0 : \mu \geqslant \mu_0$ versus $H_1 : \mu < \mu_0$ then

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \bar{x} \leqslant \mu_0 + t_{n-1,\alpha} \frac{s}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$

3. If $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$, then

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & |\bar{x} - \mu_0| \geqslant t_{n-1,\alpha/2} \frac{s}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$

t-Tests

Let $X_1, X_2, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$, where σ^2 is known.

1. If $H_0 : \mu \leqslant \mu_0$ versus $H_1 : \mu > \mu_0$, then

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \bar{x} \geqslant \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$

2. If $H_0 : \mu \geqslant \mu_0$ versus $H_1 : \mu < \mu_0$ then

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \bar{x} \leqslant \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$

3. If $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$, then

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & |\bar{x} - \mu_0| \geqslant z_\alpha \frac{\sigma}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$

t-Tests

Let $X_1, X_2, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$, where σ^2 is unknown.

1. If $H_0 : \mu \leqslant \mu_0$ versus $H_1 : \mu > \mu_0$, then

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \bar{x} \geqslant \mu_0 + t_{n-1,\alpha} \frac{s}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$

2. If $H_0 : \mu \geqslant \mu_0$ versus $H_1 : \mu < \mu_0$ then

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \bar{x} \leqslant \mu_0 + t_{n-1,\alpha} \frac{s}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$

3. If $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$, then

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & |\bar{x} - \mu_0| \geqslant t_{n-1,\alpha/2} \frac{s}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$