Exercise 1:

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim}$ `Weibull` (γ, θ), with γ known. That is,

$$f_\theta(x) = \frac{\gamma}{\theta} x^{\gamma-1} e^{-x^\gamma/\theta}, \quad x \geq 0,\ \gamma, \theta > 0,$$

Consider testing the following hypothesis problem:

$$(H) : \begin{cases} H_0 : \theta \leq \theta_0 \\ H_1 : \theta > \theta_0. \end{cases}$$

1. Assume that θ is a fixed parameter.
   (a) Find a uniformly most powerful-α (UMP-α) test of (H).
   (b) Show that the test function obtained in a) is unbiased .
   (c) Derive the corresponding uniformly most accurate unbiased (UMAU) $(1-\alpha)$ lower confidence interval.
   (d) Find the shortest-length $(1-\alpha)$ confidence interval based on the pivotal test statistc found in a). Calculate the expected length of this confidence interval.
2. Assume that the conjugate prior for θ is the inverted gamma $IG(a, b)$. Calculate the posterior probability of $H_0 : \theta \leq \theta_0$. Note $Y \sim IG(\alpha, \beta)$ if

$$f(y \| \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} y^{-(1+\alpha)} e^{-\frac{1}{\beta y}}, \quad 0 < y < \infty,\ \alpha, \beta > 0.$$

Exercise 2:

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim}$ `Logistic`$(\theta, 1)$, that is

$$f_\theta(x) = \frac{1}{2[1 + \cosh(x-\theta)]},\ x, \theta \in \mathbb{R}\cdot$$

Consider testing the following hypothesis problem:

$$(H) : \begin{cases} H_0 : \theta = 0 \\ H_1 : \theta > 0. \end{cases}$$

1. Show that the locally most powerful LMP-α test of (H) is:

$$\phi(x_1, \ldots, x_n) = \begin{cases} 1, & \sum_{i=1}^{n} \tanh\left(\frac{x_i}{2}\right) > k. \\ 0, & \text{otherwise.} \end{cases}$$

   Assume that the tests under considerations have continuously differentiable power function at $\theta = 0$ and the derivative may be taken under the integral sign.

2. To motivate the choice of $k > 0$, show that under H_0 the summands

$$\tanh(X_1/2), \ldots, \tanh(X_n/2) \overset{\perp\!\!\!\perp}{\sim} \mathcal{U}(-1, 1).$$

`Hints:` $\forall\ u \in \mathbb{R}$, $1 + \cosh(2u) = 2\cosh^2(u)$ and $\sinh(2u) = 2\sinh(u)\cosh(u)$.

Exercise 3:

Let $X_1, \ldots, X_n$ be i.i.d random variables. Consider testing the problem:

$$(H) : \begin{cases} H_0 : X_i \sim N(0, 1). \\ H_1 : X_i \sim L(0, 1). \end{cases}$$

Under H_0 and H_1, the probability densities are given by:

$$\forall\ x \in \mathbb{R}, \quad f_0(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$
$$\forall\ x \in \mathbb{R}, \quad f_1(x) = \frac{1}{2} \exp\left(-|x|\right),$$

respectively. Let $c > 0$. Consider the group of scale changes defined by:

$\mathcal{G} = \{\ \mathrm{g}_c \in F\left(\mathbb{R}^n, \mathbb{R}^n\right)\ \mid \forall\ (x_1, \ldots, x_n) \in \mathbb{R}^n,\ \mathrm{g}_c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n)\}\,.$

1. Show that the hypothesis testing problem (H) is invariant under $\mathcal{G}$.
2. Show that the following statistic is maximal invariant under $\mathcal{G}$

$$T(X_1, \ldots, X_n) = \left(X_1/X_n, \ldots, X_{n-1}/X_n\right).$$

3. (a) Show that the α-most powerful invariant (α-MPI) test of (H) is :

$$\phi(y_1, \ldots, y_{n-1}) = \begin{cases} 1, & \dfrac{\sqrt{1 + \displaystyle\sum_{i=1}^{n-1} y_i^2}}{1 + \displaystyle\sum_{i=1}^{n-1} |y_i|} > k. \\ 0, & \text{otherwise.} \end{cases}$$

where $Y_j = X_j/X_n$, $j = 1, 2, \ldots, n-1$.

(b) Equivalently, show that the α-MPI test of (H) is:

$$\phi(x_1, \ldots, x_n) = \begin{cases} 1, & \dfrac{\sqrt{\displaystyle\sum_{j=1}^{n} x_j^2}}{\displaystyle\sum_{j=1}^{n} |x_j|} > k. \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 4:

Let X and Y be two independent random variables such that
$X \sim \mathcal{B}(n, \theta_1)$ and $Y \sim \mathcal{B}(m, \theta_2)$. Consider testing the hypothesis problem:

$$(H) : \begin{cases} H_0 : \theta_1 = \theta_2 \\ H_1 : \theta_1 > \theta_2. \end{cases}$$

1. Find a sufficient statistic for $\theta = \theta_1 = \theta_2$.
2. Show that a valid p-value for testing (H) is

$$\pi(x, y) = \frac{1}{\binom{n+m}{x+y}} \sum_{k=x}^{n \wedge (x+y)} \binom{n}{k} \binom{m}{x+y-k}.$$

Exercise 5:

Assume that $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim} \mathcal{U}(0, \theta)$, with $\theta > 0$.

1. Find the MLE for θ and study its consistency.
2. Let $\epsilon > 0$. Use the Chebyshev's inequality to show that a $(1 - 1/\epsilon^2)$ confidence interval for θ is:

$$I\left(X_{(n)}\right) = \left(X_{(n)} - \epsilon X_{(n)} \frac{\sqrt{2}}{\sqrt{(n+1)(n+2)}}, \quad X_{(n)} + \epsilon X_{(n)} \frac{\sqrt{2}}{\sqrt{(n+1)(n+2)}} \right).$$