Exercise 1:

Let X be one observation from Logistic$(\theta, 1)$, with $\theta \in \mathbb{R}$.

1. Show that this family has a monotone likelihood ratio property.
2. Find the most powerful size α test of

$$(H) : \begin{cases} H_0 : \theta = 0 \\ H_1 : \theta = 1. \end{cases}$$

   For $\alpha = 0.2$, find the size of the type-II error.

3. Show that the test in part (2) is UMP size α for testing

$$(H) : \begin{cases} H_0 : \theta \leq 0 \\ H_1 : \theta > 0. \end{cases}$$

Exercise 2:

Let $X_1, X_2, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim}$ Pareto$(1, \theta)$. Let μ denote its mean.

1. Express μ in terms of θ.
2. Find a UMP test for testing

$$(H) : \begin{cases} H_0 : \mu = \mu_0. \\ H_1 : \mu < \mu_0. \end{cases}$$

3. Find the distribution of the test statistic found in (2) under H_0.

Exercise 3:

Let $\Theta, \mathcal{X} \subseteq \mathbb{R}$. Suppose that $\mathcal{G} = \{g_\theta(x) : \theta \in \Theta, x \in \mathcal{X}\}$ be the family of density functions such that $\forall\, x \in \mathcal{X}, \theta \in \Theta$,

$$\begin{cases} g_\theta(x) > 0. \\ \frac{\partial^2}{\partial\theta\partial x} \ln g_\theta(x) < +\infty. \end{cases}$$

1. Show that $\mathcal{G}$ has a monotone likelihood ratio property in x is equivalent to one of the following conditions:

$$i).\ \forall\, x \in \mathcal{X}, \theta \in \Theta,\ \frac{\partial^2}{\partial\theta\partial x} \ln g_\theta(x) \geq 0.$$

$$ii).\ \forall\, x \in \mathcal{X}, \theta \in \Theta,\ g_\theta(x) \frac{\partial^2}{\partial\theta\partial x} g_\theta(x) \geq \frac{\partial}{\partial\theta} g_\theta(x) \frac{\partial}{\partial x} g_\theta(x).$$

2. If $X \sim \mathcal{N}(\sqrt{\theta}, 1)$, then $Z^2 \sim \mathcal{X}_1^2(\theta)$. Show that $\{\mathcal{X}_1^2(\theta)\}$ has MLR in x.

Exercise 4:

Let X be a random variable whose pmf under $H_i,\ i=\overline{1,2}$ is given by

x	1	2	3	4	5	6	7
$f_0(x)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f_1(x)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

1. Find the most powerful test for H_0 versus H_1 with size $\alpha = 0.04$.
2. Compute the probability of type II error for this test.

Exercise 5:

Let $X_1, \ldots, X_n$ be a sample from $\mathcal{U}(\theta, \theta+1)$ distribution. To test

$$(H):\begin{cases} H_0 : \theta = 0 \\ H_1 : \theta > 0, \end{cases}$$

Consider the following statistical test function ϕ such that:

$$\phi(x_1, x_2, \ldots, x_n) = \begin{cases} 1, & \max(x_1, \ldots, x_n) > 1 \text{ or } \min(x_1, \ldots, x_n) > k \\ 0, & \text{otherwise} . \end{cases}$$

1. Find the constant k that makes ϕ is a size α test.
2. Find the power function of the test ϕ
3. Prove that ϕ is a UMP size α test.
4. Find the values of n and k so that the UMP 0.10 level test will have power at least 0.8 if $\theta > 1$.

Exercise 6:

Let $X_1, \ldots, X_n \overset{\perp\!\!\!\perp}{\sim} \mathcal{N}(\mu_1, \sigma_1^2)$, and $Y_1, \ldots, Y_m \overset{\perp\!\!\!\perp}{\sim} \mathcal{N}(\mu_2, \sigma_2^2)$. Consider testing

$$(H):\begin{cases} H_0 : \mu_1 = \mu_2 \\ H_1 : \mu_1 \neq \mu_2 \end{cases}$$

with the assumption that $\sigma_1^2 = \sigma_2^2 = \sigma^2$.

1. Find the likelihood ratio test of (H).

2. Show that the LRT can be based on the statistic

$$T = \frac{\bar{X} - \bar{Y}}{\sqrt{S_P^2(\frac{1}{n} + \frac{1}{m})}}, \text{ where}$$

$$S_P^2 = \frac{1}{(n+m-2)} \left(\sum_{i=1}^{n} (X_i - \bar{X})^2 + \sum_{i=1}^{m} (Y_i - \bar{Y})^2 \right).$$

3. Find the distribution of T when H_0 is true.