Exercise 1:

Let $X \sim \mathcal{L}(\theta, \sigma) \Rightarrow f(x/\theta,\sigma) = \frac{1}{2\sigma} e^{-\frac{|x-\theta|}{\sigma}}$, $x \in \mathbb{R}$, $\theta \in \mathbb{R}$, $\sigma > 0$.

Assume that the scale parameter σ is known.

1)- Show that this family of pdfs has MLR in x:

Let $\theta_2 > \theta_1$. The likelihood ratio is

$$\frac{f(x/\theta_2)}{f(x/\theta_1)} = e^{\frac{1}{\sigma}\left[|x-\theta_1| - |x-\theta_2|\right]}.$$

It follows from the definition of $u \mapsto |u| = \begin{cases} u, & u \geq 0 \\ -u, & u < 0 \end{cases}$

x	$-\infty$	θ_1	θ_2	$+\infty$
$\lvert x-\theta_1\rvert$	$-x+\theta_1$ 0	$x-\theta_1$		$x-\theta_1$
$-\lvert x-\theta_2\rvert$	$x-\theta_2$	$x-\theta_2$ 0		$-x+\theta_2$
$\lvert x-\theta_1\rvert - \lvert x-\theta_2\rvert$	$\theta_1-\theta_2$	$2x-(\theta_1+\theta_2)$		$\theta_2-\theta_1$

Therefore, the likelihood ratio is defined as follows:

$$\frac{f(x/\theta_2)}{f(x/\theta_1)} = \begin{cases} e^{\frac{1}{\sigma}(\theta_1-\theta_2)}, & x < \theta_1 \\ e^{\frac{1}{\sigma}(2x-\theta_1-\theta_2)}, & \theta_1 \leq x < \theta_2 \\ e^{\frac{1}{\sigma}(\theta_2-\theta_1)} & x \geq \theta_2. \end{cases}$$

But $\forall \theta_1 < \theta_2$, $x \longmapsto \frac{f(x/\theta_2)}{f(x/\theta_1)}$ is continuous and nondecreasing.

Thus, this family of pdfs has MLR in x.

2)- Find a UMP-α test for (H): $\begin{cases} H_0 : \theta \leq \theta_0 \\ H_1 : \theta > \theta_0 . \end{cases}$

Since $\{\mathcal{L}(\theta,\sigma) : \theta \in \mathbb{R}\}$ has MLR in x, then the Karlin-Rubin theorem yields that

$$\varphi(x) = \begin{cases} 1, & x > k \\ 0, & o/w, \end{cases}$$

is a UMP size α test for (H).

Since $E_{\theta_0}[\varphi(X)] = \alpha \iff P_{\theta_0}(X > k) = \alpha$, then

$*$ $\theta_0 > k \Rightarrow \int_k^\infty e^{-\frac{|x-\theta|}{\sigma}} dx = 2\sigma\alpha$

$$\int_k^{\theta_0} e^{\frac{(x-\theta_0)}{\sigma}} dx + \int_{\theta_0}^\infty e^{-\frac{(x-\theta_0)}{\sigma}} dx = 2\sigma\alpha$$

$$\left[\sigma e^{\frac{x-\theta_0}{\sigma}}\right]_k^{\theta_0} + \left[-\sigma e^{-\frac{(x-\theta_0)}{\sigma}}\right]_{\theta_0}^\infty = 2\sigma\alpha$$

$$\sigma - \sigma e^{\frac{k-\theta_0}{\sigma}} + \sigma = 2\sigma\alpha$$

$$2\sigma(1-\alpha) = \sigma e^{\frac{k-\theta_0}{\sigma}}$$

$$k = \sigma \ln[2(1-\alpha)] + \theta_0 .$$

$*$ $\theta_0 < k \Rightarrow \int_k^\infty e^{-\frac{|x-\theta_0|}{\sigma}} dx = 2\sigma\alpha$

$$\Rightarrow \int_k^\infty e^{\frac{-x+\theta_0}{\sigma}} dx = 2\sigma\alpha$$

$$\Rightarrow \left[-\sigma e^{\frac{-x+\theta_0}{\sigma}}\right]_k^\infty = 2\sigma\alpha$$

$$\Rightarrow \sigma e^{\frac{-k+\theta_0}{\sigma}} = 2\sigma\alpha$$

$$\Rightarrow e^{-\frac{k+\theta_0}{\sigma}} = 2\alpha$$

$$\Rightarrow k = \theta_0 - \sigma \ln(2\alpha)$$

All in all, $\varphi(x) = \begin{cases} 1, & x > k \\ 0, & o/w \end{cases}$ is a UMP size α test of (H),

with $k = \begin{cases} \theta_0 + \sigma \ln[2(1-\alpha)], & \text{if } \theta_0 > k \\ \theta_0 - \sigma \ln(2\alpha), & \text{if } \theta_0 < k. \end{cases}$

Exercise 2:

Consider testing (H): $\begin{cases} H_0 : X \sim \mathcal{N}(0,1) \\ H_1 : X \sim \mathcal{L}(0,1). \end{cases}$

1) Find a most powerful size α test of (H).

The likelihood ratio is given by:

$$\lambda(x) = \frac{f_1(x)}{f_0(x)} = \sqrt{\frac{\pi}{2}}\, e^{-|x| + \frac{x^2}{2}} = \sqrt{\frac{\pi}{2}}\, e^{\frac{1}{2}\left[|x|^2 - 2|x|\right]}$$
$$= \sqrt{\frac{\pi e}{2}}\, e^{\frac{1}{2}(|x|-1)^2}$$

The Neyman-Pearson lemma yields that

$$\varphi(x) = \begin{cases} 1, & \lambda(x) > k \quad (k>0) \\ 0, & \text{o/w.} \end{cases}$$

is an MP test of (H) such that $E[\varphi(X)] = \alpha$, under $H_0 : X \sim \mathcal{N}(0,1)$.

Notice that $x \longmapsto g(x) = \sqrt{\frac{\pi e}{2}}\, e^{\frac{1}{2}x^2}$ behaves as follows:

x	$-\infty$		0		$+\infty$
$g'(x)$		$-$	0	$+$	
$g(x)$	$+\infty$	$\searrow$	$\sqrt{\frac{\pi e}{2}}$	$\nearrow$	$+\infty$

Therefore, $x \longmapsto g(x)$ is nondecreasing for $x \geq 0$.

Thus, $||x|-1| \longmapsto g(||x|-1|) = \lambda(x)$ is nondecreasing.

Hence, $\lambda(x) > k \iff ||x|-1| > c$.

So, $\varphi(x) = \begin{cases} 1, & ||x|-1| > c \\ 0, & \text{o/w.} \end{cases}$ is an MP test of (H) such that $E[\varphi(X)] = \alpha$, under $H_0 : X \sim \mathcal{N}(0,1)$.

Since $E[\varphi(X)]=\alpha$, under $H_0: X \sim \mathcal{N}(0,1)$, then

$$P(||X|-1|>c)=\alpha \Rightarrow P(|X|<1-c)+P(|X|>1+c)=\alpha, \quad X\sim\mathcal{N}(0,1)$$

$$\Rightarrow P(|X|<1-c)+1-P(|X|\le 1+c)=\alpha$$

$$\Rightarrow 2\phi(1-c)+1-2\phi(1+c)=\alpha$$

$$\Rightarrow \phi(1+c)-\phi(1-c)=\frac{1-\alpha}{2} \quad (*)$$

Let $c=z_{\alpha}$ be the solution of the equation (*).

2)- <u>Find the power of φ against H_1</u>:

By definition, the power of φ against H_1 is

$$\beta_{\varphi}(\theta)=E[\varphi(X)]=P(||X|-1|>c), \text{ under } H_1: X\sim\mathcal{L}(0,1).$$

$$=P(|X|<1-c)+1-P(|X|\le 1+c)$$

$$=\frac{1}{2}\int_{c-1}^{1-c} e^{-|x|}\,dx+1-\frac{1}{2}\int_{-(1+c)}^{1+c} e^{-|x|}\,dx$$

$$=1+e^{-(1+c)}-e^{-(1-c)}, \quad \text{with } c=z_{\alpha}.$$

Exercise 3:

$X_1, \dots, X_n \overset{iid}{\sim} \mathcal{B}(1,\theta)$.

Find a UMP-α test for testing $(H): \begin{cases} H_0: \theta = \theta_0 \\ H_1: \theta < \theta_0 \end{cases}$

Notice $(H) \Leftrightarrow (H'): \begin{cases} H_0: \theta = \theta_0 \\ H_1: \theta = \theta_1 \end{cases} \quad (\theta_1 < \theta_0)$.

Under H_0, $f(x \| \theta_0) = \prod_{i=1}^{n} f(x_i \| \theta_0) = \theta_0^{\sum_{i=1}^n x_i} (1-\theta_0)^{n-\sum_{i=1}^n x_i}$

Under H_1, $f(x \| \theta_1) = \prod_{i=1}^{n} f(x_i \| \theta_1) = \theta_1^{\sum_{i=1}^n x_i} (1-\theta_1)^{n-\sum_{i=1}^n x_i}$

So, the likelihood ratio is

$$\lambda(\underline{x}) = \frac{f(\underline{x} \| \theta_1)}{f(\underline{x} \| \theta_0)} = \left(\frac{\theta_1}{\theta_0}\right)^{\sum_{i=1}^n x_i} \left(\frac{1-\theta_1}{1-\theta_0}\right)^{n-\sum_{i=1}^n x_i}$$

The Neyman-Pearson lemma yields that

$$\varphi(\underline{x}) = \begin{cases} 1, & \lambda(\underline{x}) > k \\ \gamma, & \lambda(\underline{x}) = k \\ 0, & \lambda(\underline{x}) < k, \end{cases} \quad (k>0)$$

is an MP-test / $E_{\theta_0}[\varphi(\underline{X})] = \alpha$.

Since $\theta_1 < \theta_0$, then $\frac{\theta_1}{\theta_0} < 1$ and $\frac{1-\theta_0}{1-\theta_1} < 1$.

Therefore, $\underline{x} \longmapsto \lambda(\underline{x}) = \left(\frac{1-\theta_1}{1-\theta_0}\right)^n \left[\frac{\theta_1}{\theta_0}\left(\frac{1-\theta_0}{1-\theta_1}\right)\right]^{\sum_{i=1}^n x_i}$ is

nonincreasing. In fact, $x \longmapsto a^x$ is nonincreasing $\Leftrightarrow a<1$.

Thus, an MP-α test of (H') is

$$\varphi(\underline{x}) = \begin{cases} 1, & \sum_{i=1}^n x_i < c \\ \gamma, & \sum_{i=1}^n x_i = c \\ 0, & \sum_{i=1}^n x_i > c. \end{cases}$$

Since $E_{\theta_0}[\varphi(\underline{X})]=\alpha$, for $c \in \{0,1,\dots,n\}$, we have

$$P_{\theta_0}\left(\sum_{i=1}^{n} X_i < c\right) + \gamma P_{\theta_0}\left(\sum_{i=1}^{n} X_i = c\right) = \alpha$$

$$\sum_{i=0}^{c-1} \binom{n}{i} \theta_0^i (1-\theta_0)^{n-i} + \gamma \binom{n}{c} \theta_0^c (1-\theta_0)^{n-c} = \alpha \text{ , as } \sum_{i=1}^{n} X_i \sim \mathcal{B}(n,\theta_0).$$

under H_0.

But, this MP size α test φ is independent of the choice of θ_1, as long as $\theta_1 < \theta_0$, ie φ remains an MP size α test against any $\theta < \theta_0$.

Thus, φ is UMP size α test.

Exercise 4:

Let $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{U}(0,\theta)$, $\theta > 0$.

Consider testing $(H): \begin{cases} H_0: \theta = \theta_0 \\ H_1: \theta \neq \theta_0 \end{cases}$

<u>Find an LRT of (H):</u>

- By definition, $L(\theta \| \underline{x}) = \frac{1}{\theta^n} \mathbb{1}_{[x_{(n)}, \infty)}(\theta)$.

  Since $\theta \longmapsto \frac{1}{\theta^n}$ is nonincreasing over $[x_{(n)}, \infty)$, then it attains its maximum at $x_{(n)}$.

  Therefore, $\hat{\theta}_{MLE} = X_{(n)}$.

- Under $H_0: \theta = \theta_0$, $\hat{\theta}_0 = \theta_0$. It follows that:

  $$L(\hat{\theta}_0 \| \underline{x}) = \frac{1}{\theta_0^n} \mathbb{1}_{[x_{(n)}, +\infty)}(\theta_0) = \begin{cases} \frac{1}{\theta_0^n}, & x_{(n)} \leq \theta_0 \\ 0, & x_{(n)} > \theta_0 \end{cases}$$

  So, the LR test statistic is:

  $$\lambda(\underline{x}) = \frac{L(\theta_0 \| \underline{x})}{L(x_{(n)} \| \underline{x})} = \begin{cases} \left(\frac{x_{(n)}}{\theta_0}\right)^n, & x_{(n)} \leq \theta_0 \\ 0, & x_{(n)} > \theta_0. \end{cases}$$

  Thus, a size α LRT ϕ rejects $H_0: \theta = \theta_0$ if $\lambda(\underline{x}) < k$, $k \in (0,1)$. Notice that $\lambda(\underline{x}) < k \Leftrightarrow x_{(n)} > \theta_0 \cap x_{(n)} \leq c$.

  Hence, $\phi(\underline{x}) = \begin{cases} 1, & x_{(n)} > \theta_0 \text{ or } x_{(n)} \leq c \\ 0, & \text{otherwise} \end{cases}$ is a size α LRT of (H).

  Since $E_{\theta_0}[\phi(\underline{X})] = \alpha \Leftrightarrow P_{\theta_0}\left[(X_{(n)} > \theta_0) \cup (X_{(n)} \leq c)\right] = \alpha$

  $\Leftrightarrow P_{\theta_0}(X_{(n)} \leq c) = \alpha$, since under H_0, $P_{\theta_0}(X_{(n)} > \theta_0) = 0$

  $\Rightarrow c = \theta_0 \sqrt[n]{\alpha}$.

So, the size α LRT of (H) is

$$\phi(\underline{x}) = \begin{cases} 1, & x_{(n)} > \theta_0 \text{ or } x_{(n)} \leq \theta_0 \sqrt[n]{\alpha} \\ 0, & \text{otherwise.} \end{cases}$$

2)- <u>Find the power function of the test:</u>

Recall that $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{U}(0,\theta)$, $\theta > 0$.

By definition, $\beta_\phi(\theta) = E_\theta[\phi(\underline{X})]$, $\forall \theta > 0$

$$= P_\theta\left[(X_{(n)} > \theta_0) \cup (X_{(n)} \leq \theta_0 \sqrt[n]{\alpha})\right]$$

$$= P_\theta\left(X_{(n)} \leq \theta_0 \sqrt[n]{\alpha}\right) + P_\theta\left(X_{(n)} > \theta_0\right)$$

$$= \begin{cases} 1, & 0 < \theta < \theta_0 \sqrt[n]{\alpha} \\ P_\theta\left(X_{(n)} \leq \theta_0 \sqrt[n]{\alpha}\right), & \theta_0 \sqrt[n]{\alpha} \leq \theta < \theta_0 \\ P_\theta\left(X_{(n)} \leq \theta_0 \sqrt[n]{\alpha}\right) + P_\theta\left(X_{(n)} > \theta_0\right), & \theta > \theta_0 \end{cases}$$

$$= \begin{cases} 1, & 0 < \theta < \theta_0 \sqrt[n]{\alpha} \\ \alpha\left(\frac{\theta_0}{\theta}\right)^n, & \theta_0 \sqrt[n]{\alpha} \leq \theta < \theta_0 \\ 1 - (1-\alpha)\left(\frac{\theta_0}{\theta}\right)^n, & \theta > \theta_0 . \end{cases}$$